WENTWORTH INSTITUTE OF TECHNOLOGY

SAN DIEGO STATE UNIVERSITY

The Role of Deformation on the Stellar Structure of Non-Rotating Neutron Stars

Omair Zubairi
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Wentworth Institute of Technology

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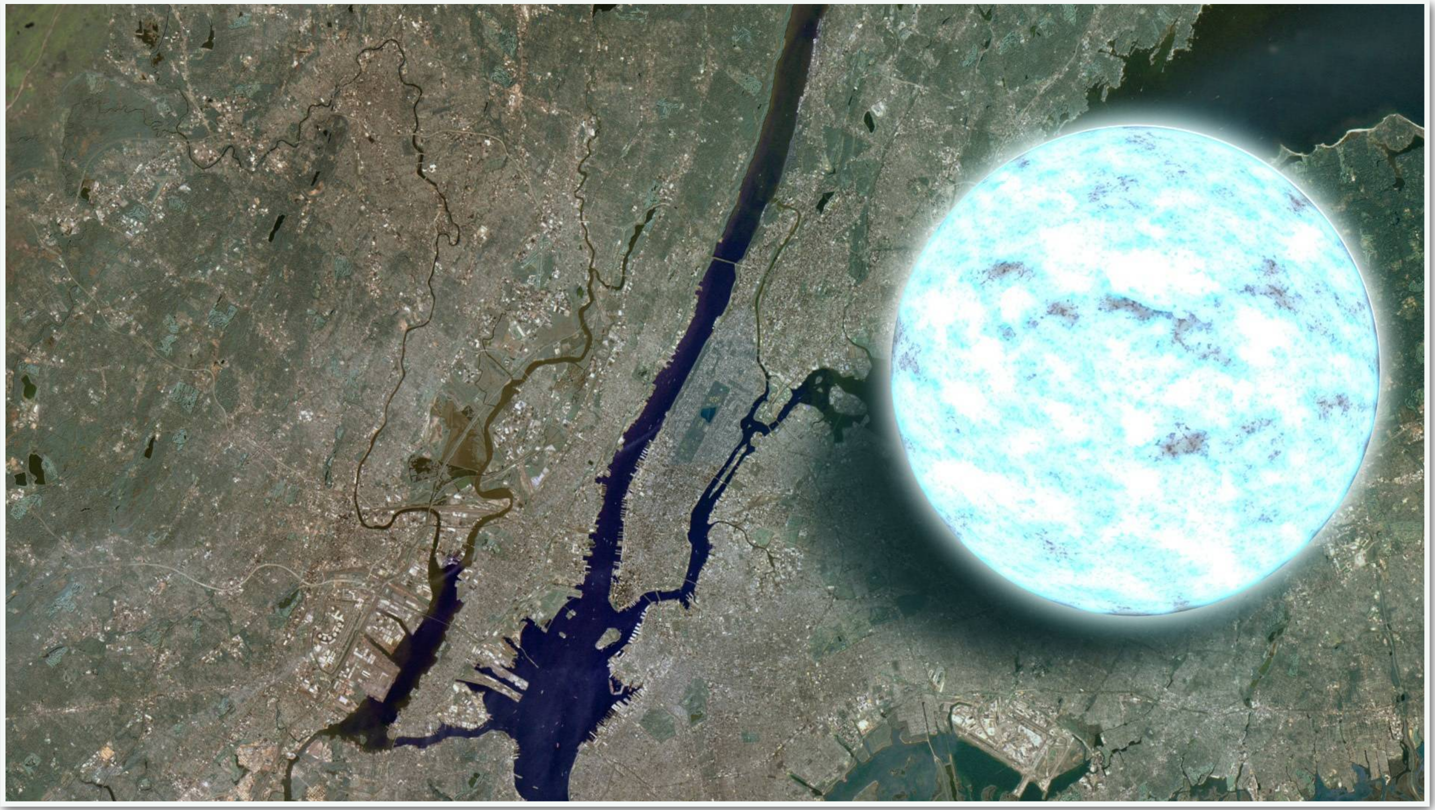
San Diego State University



CSQCD VII 2018, New York, NY

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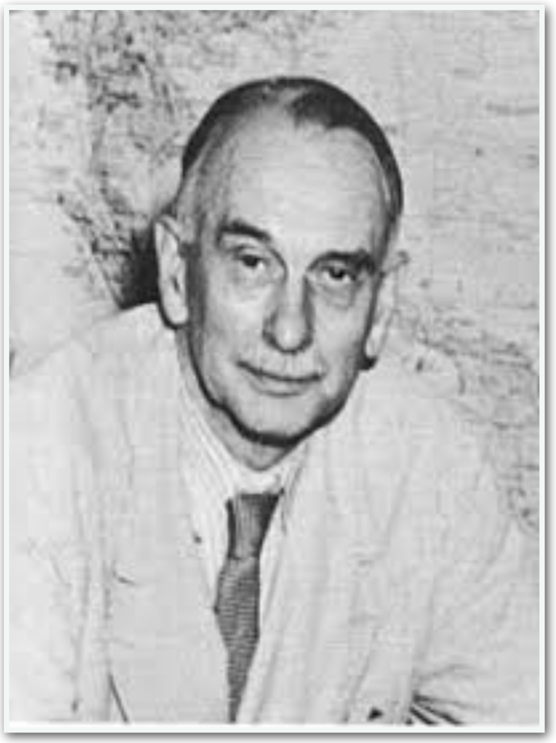

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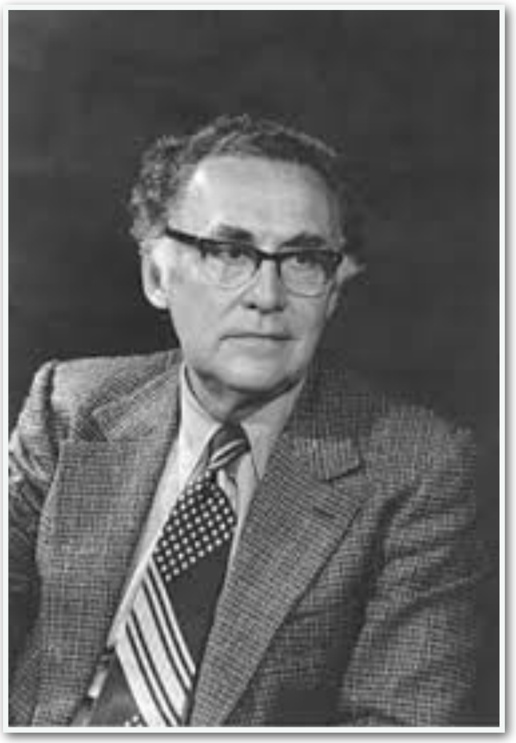

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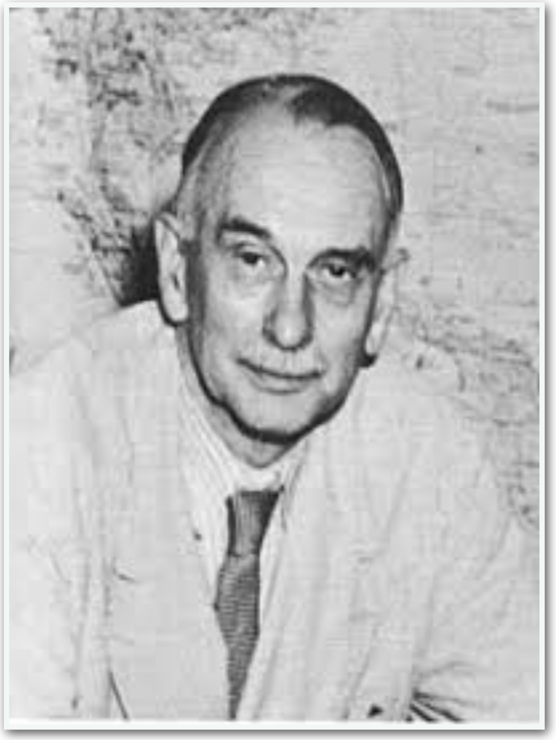

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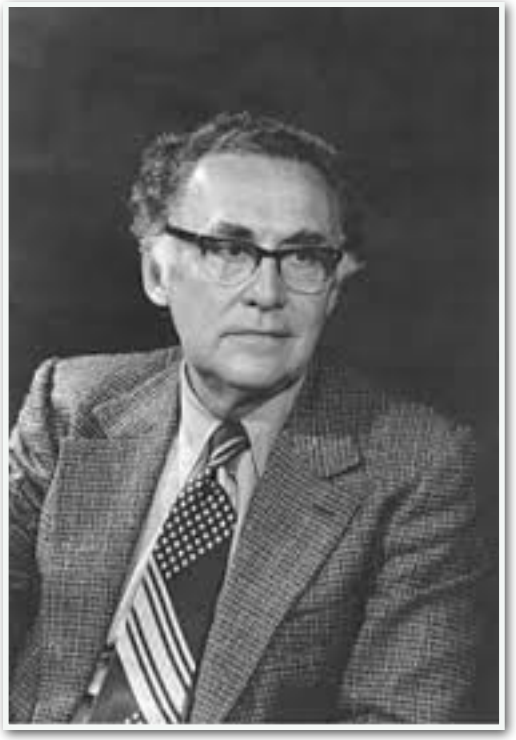

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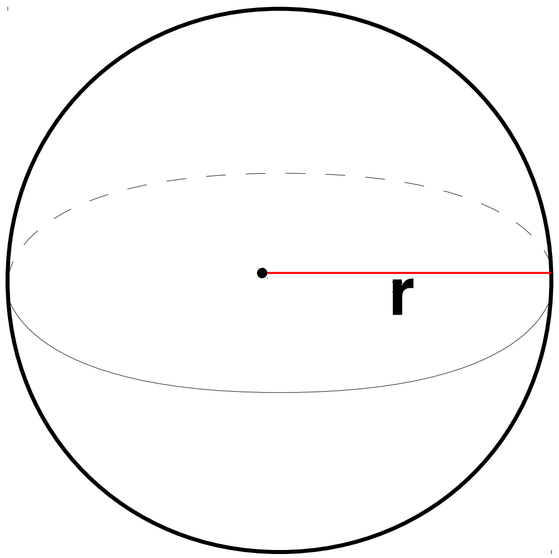

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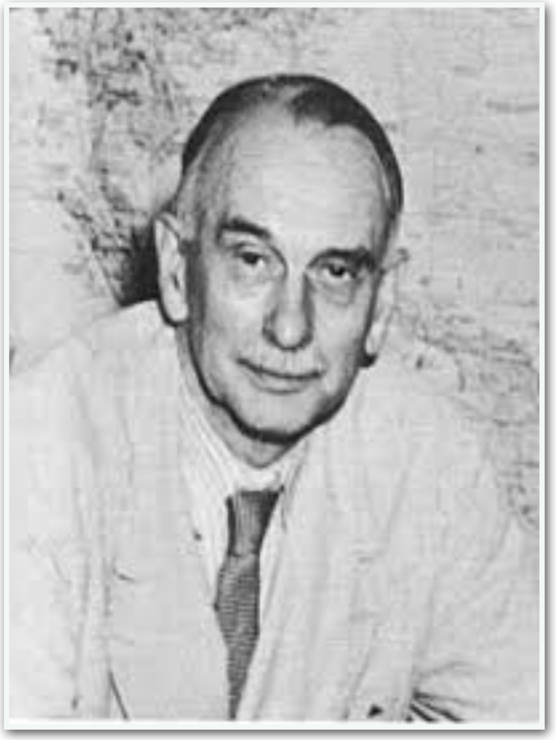

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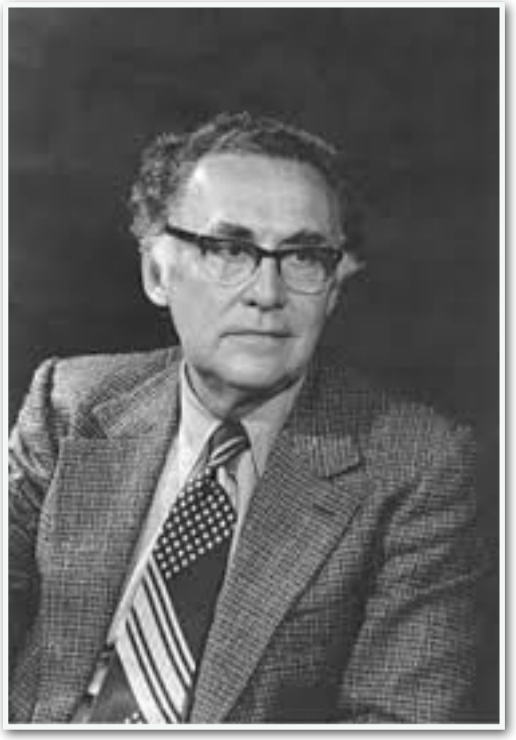

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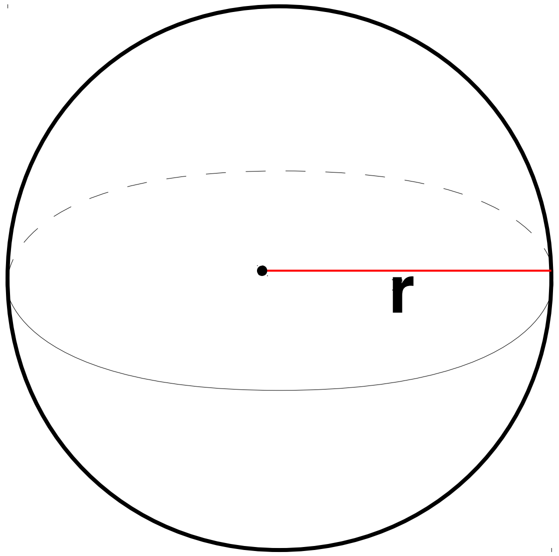

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$$ds^2 = \mathrm{e}^{2\Phi}\mathrm{dt}^2 - \mathrm{e}^{2\Lambda}\mathrm{dr}^2 - \mathrm{r}^2\mathrm{d}\theta^2 - \mathrm{r}^2\sin^2(\theta)\mathrm{d}\phi^2$$

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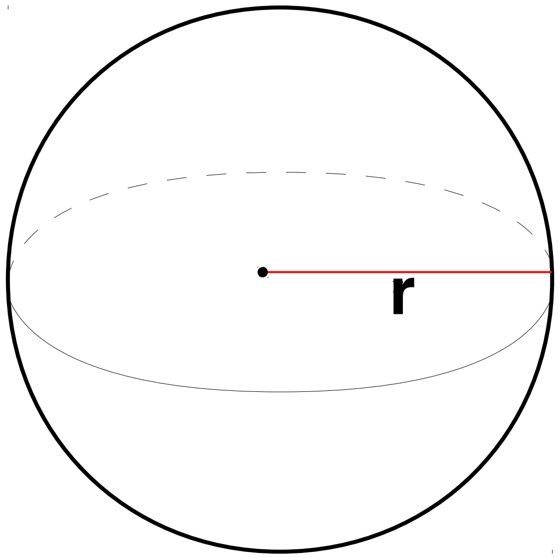

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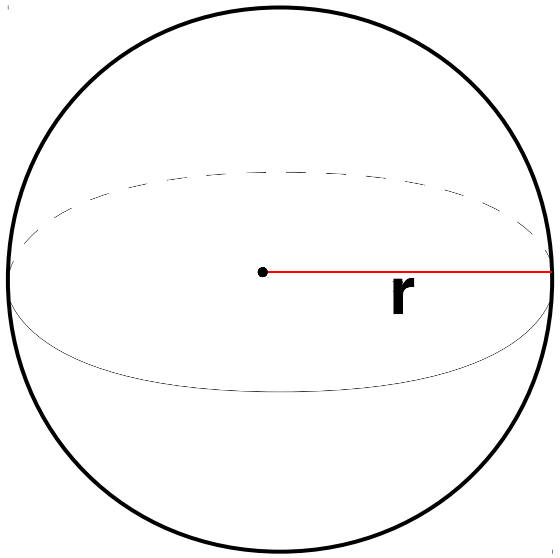

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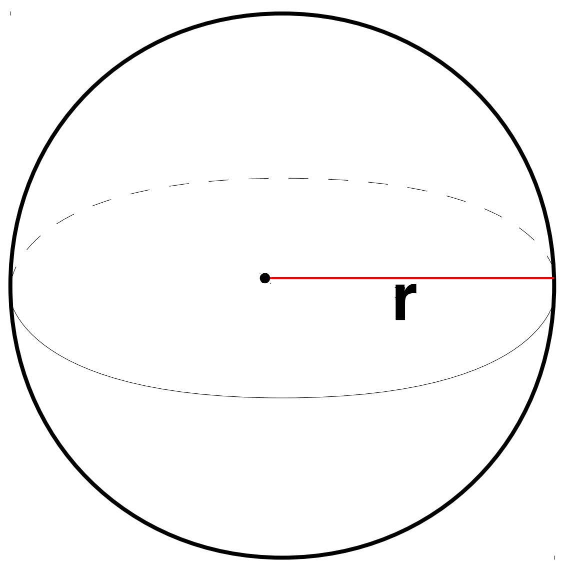

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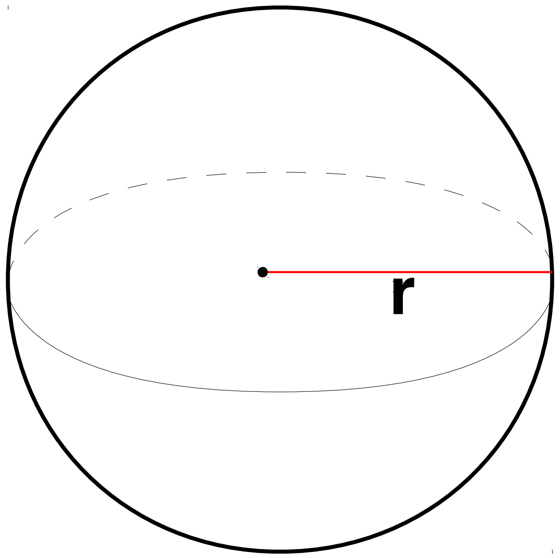

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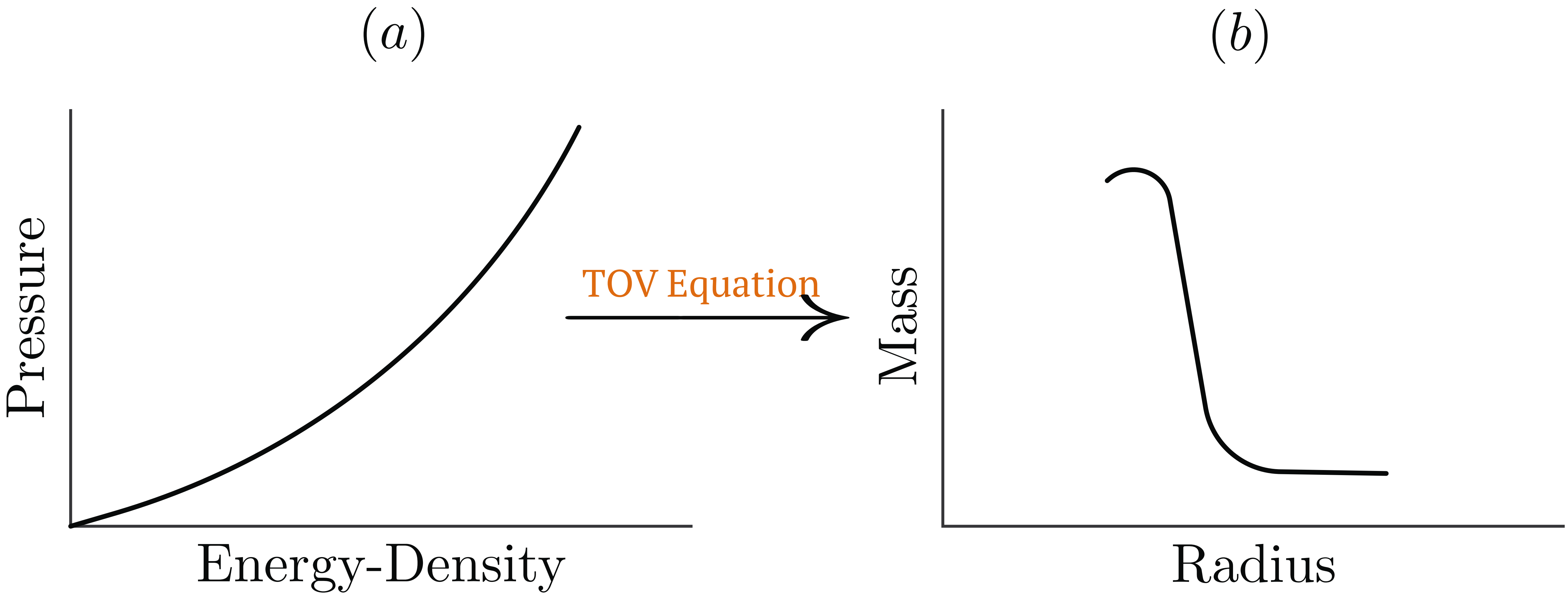

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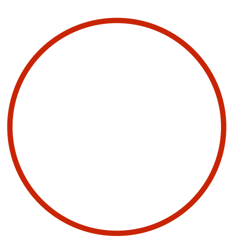

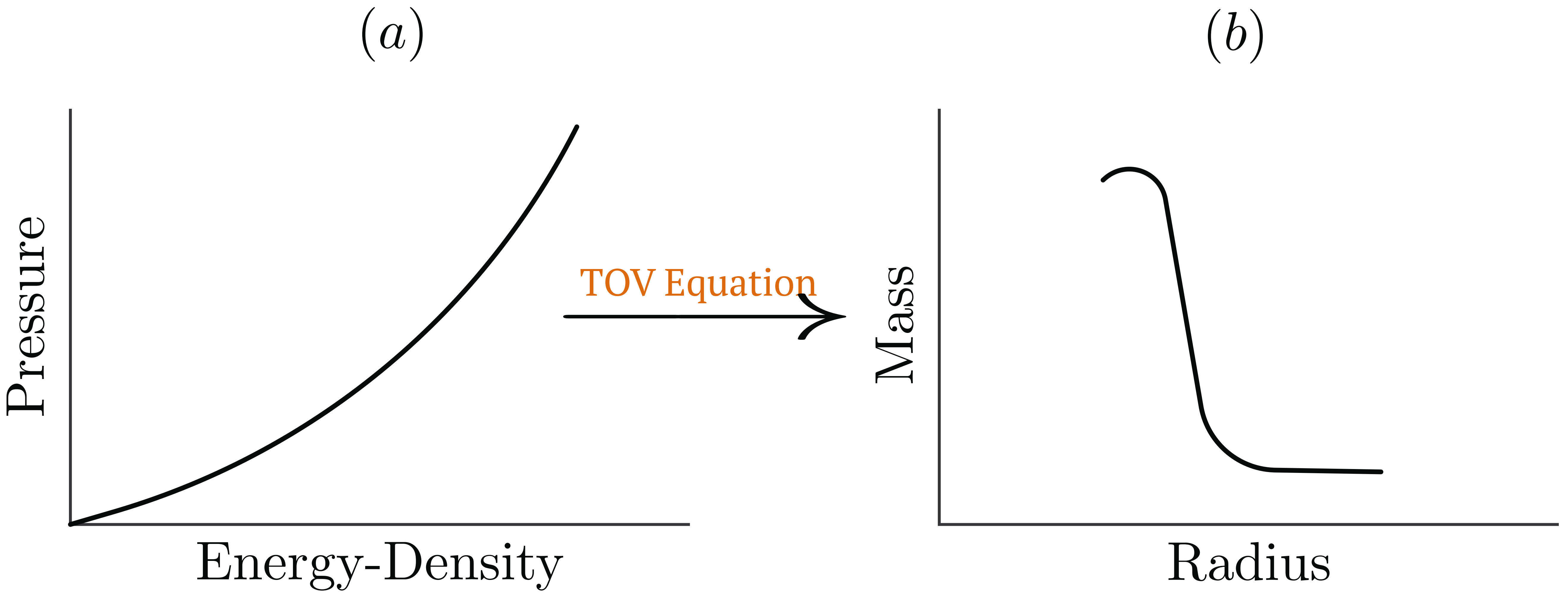

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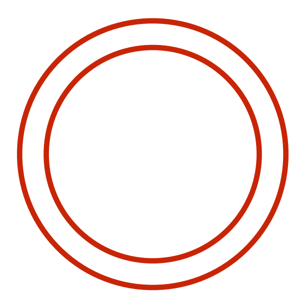

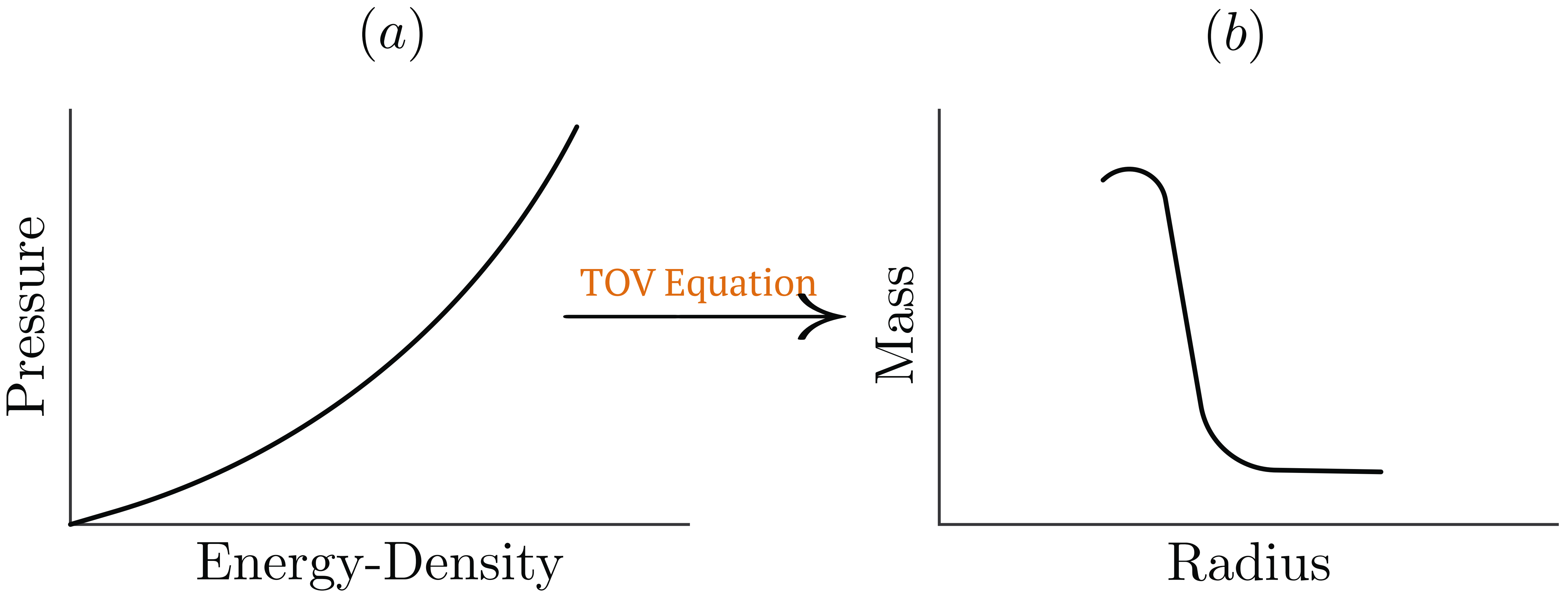

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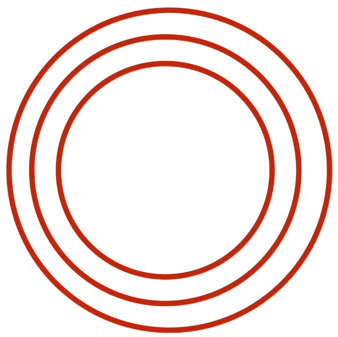

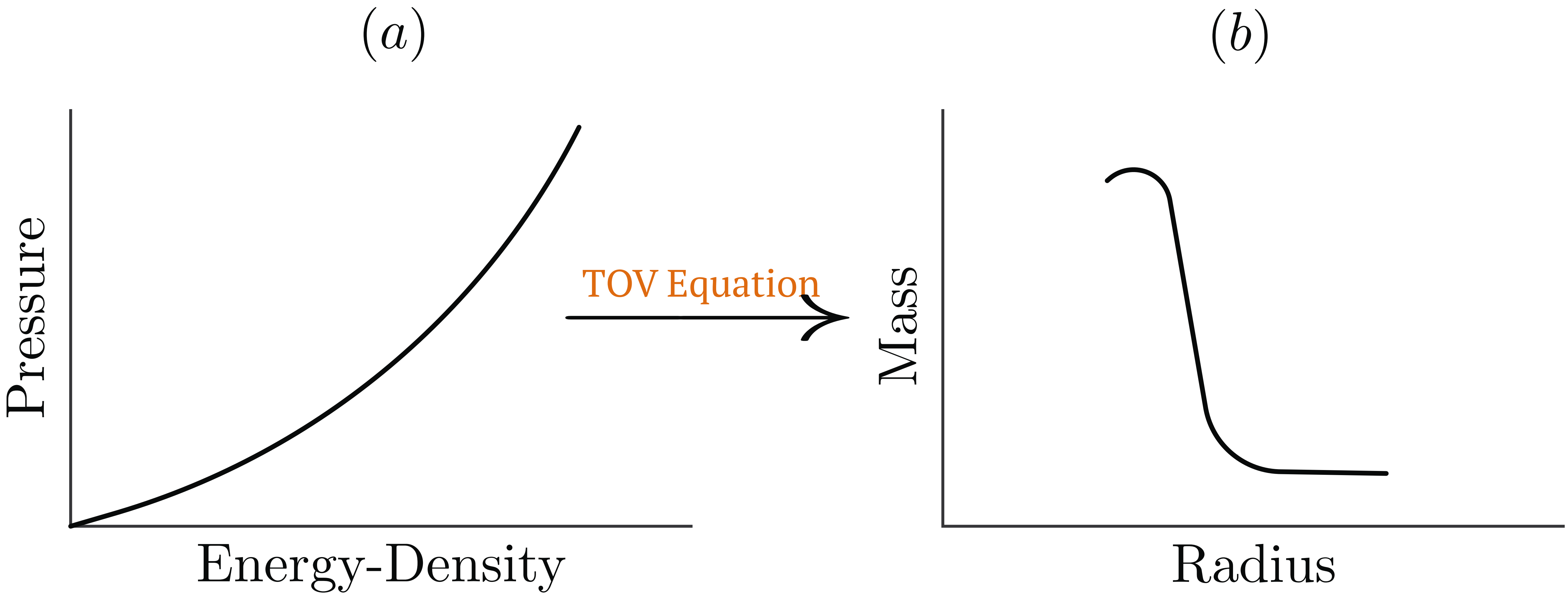

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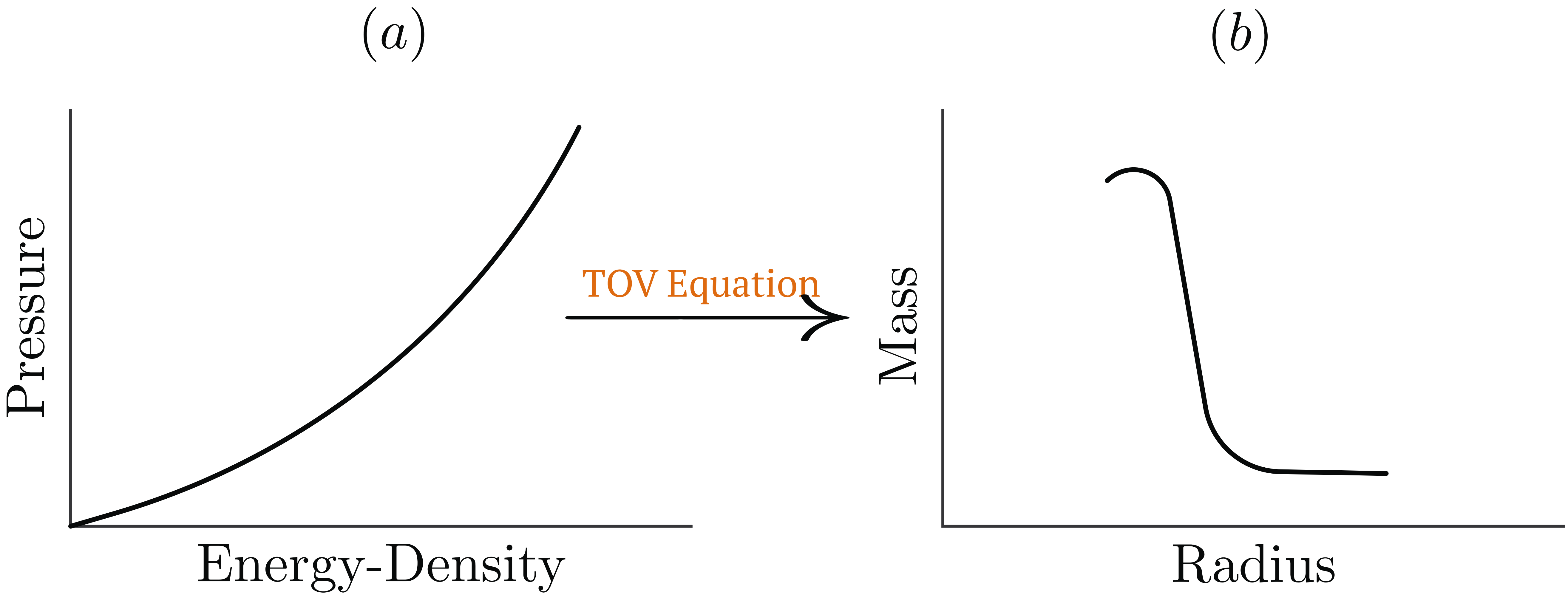

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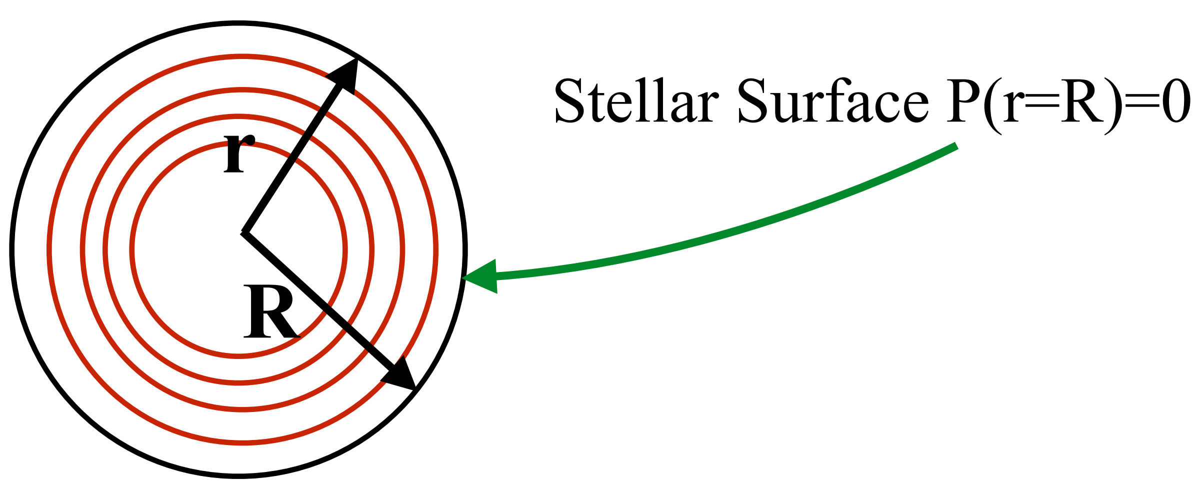

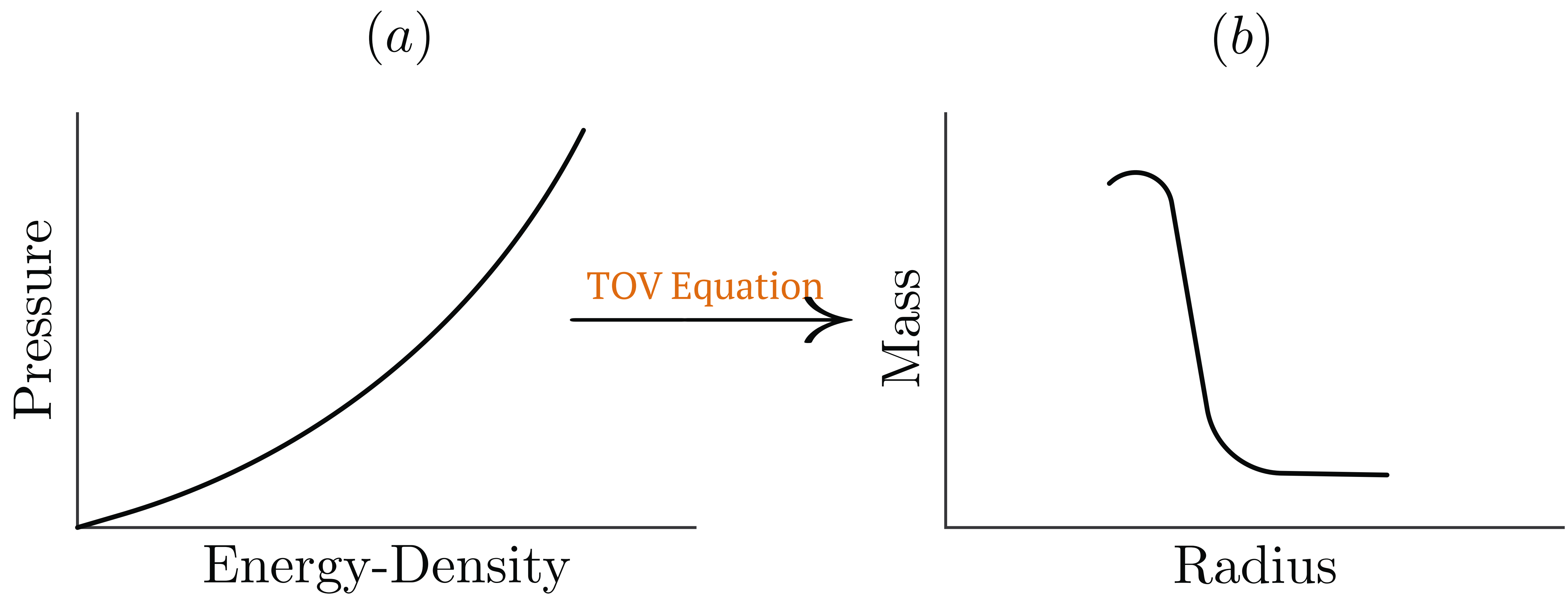

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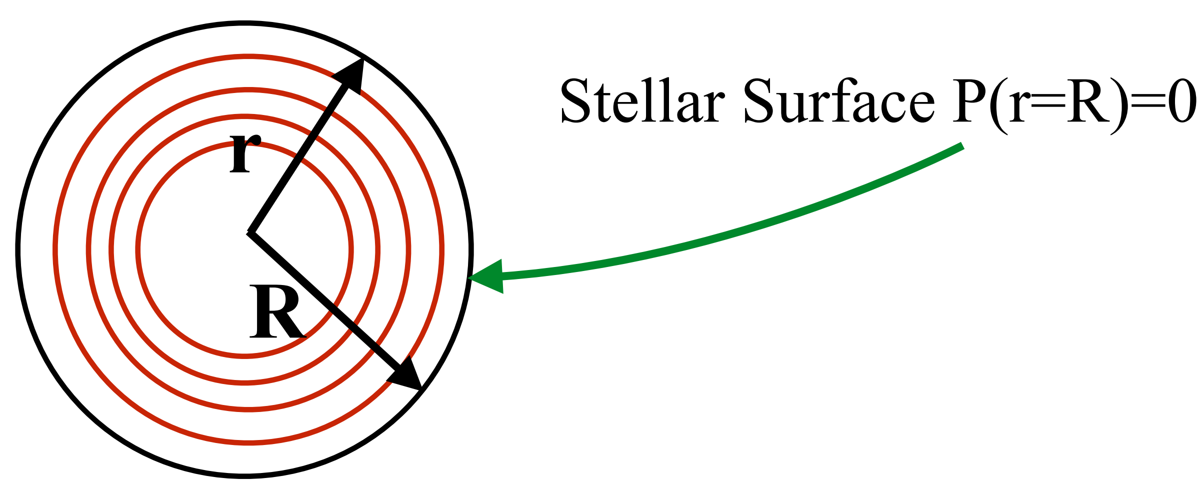

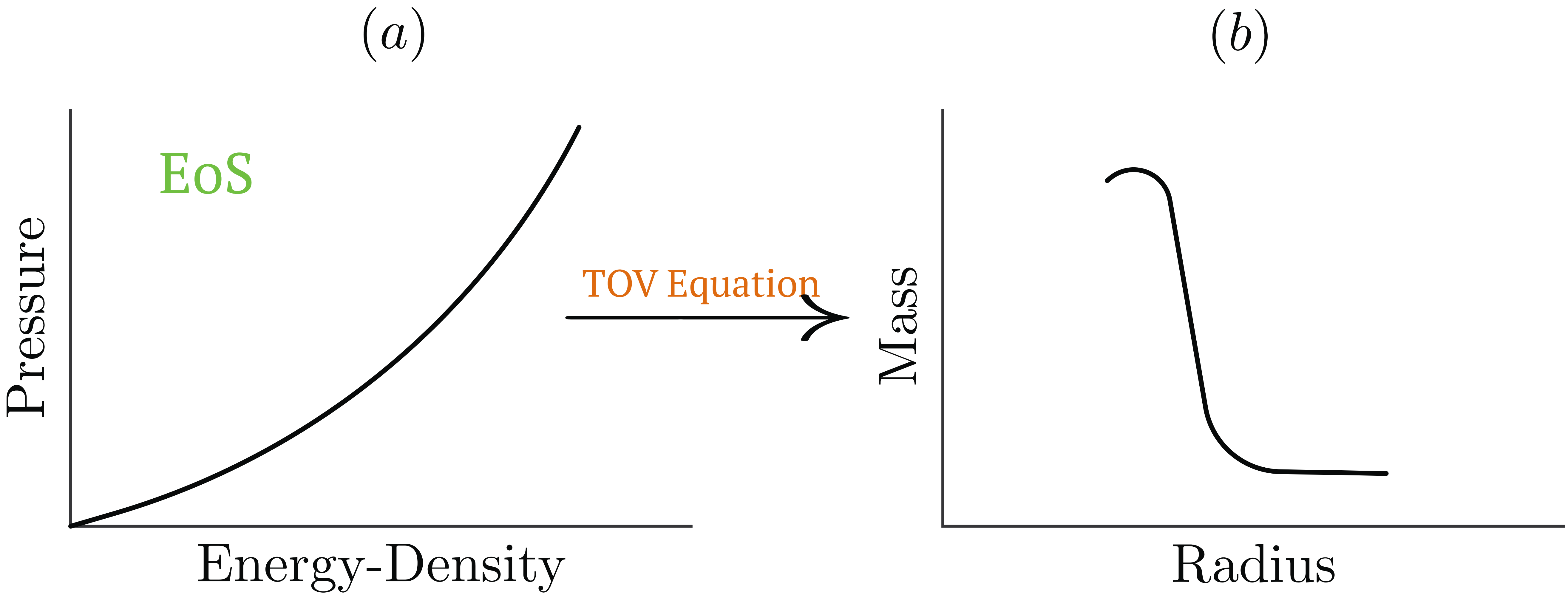

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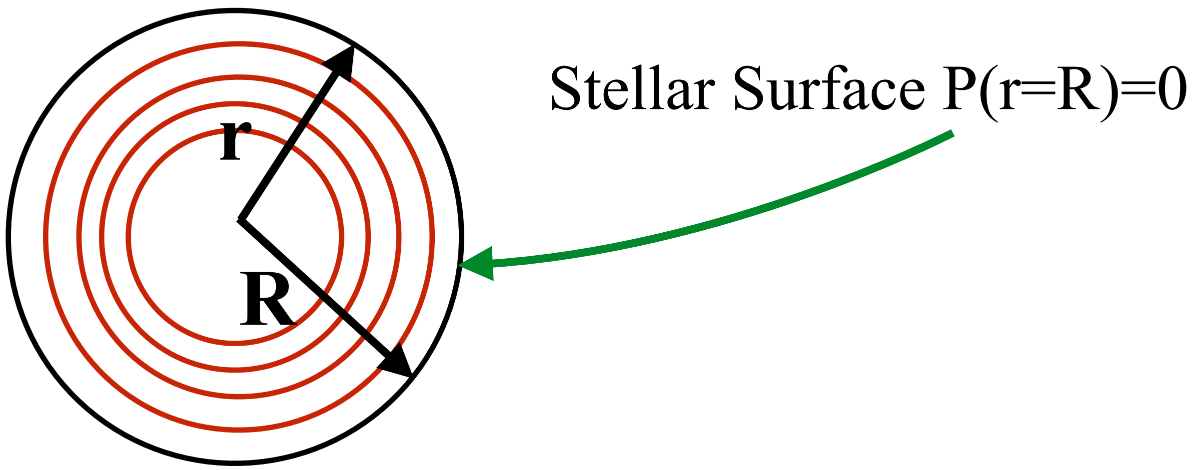

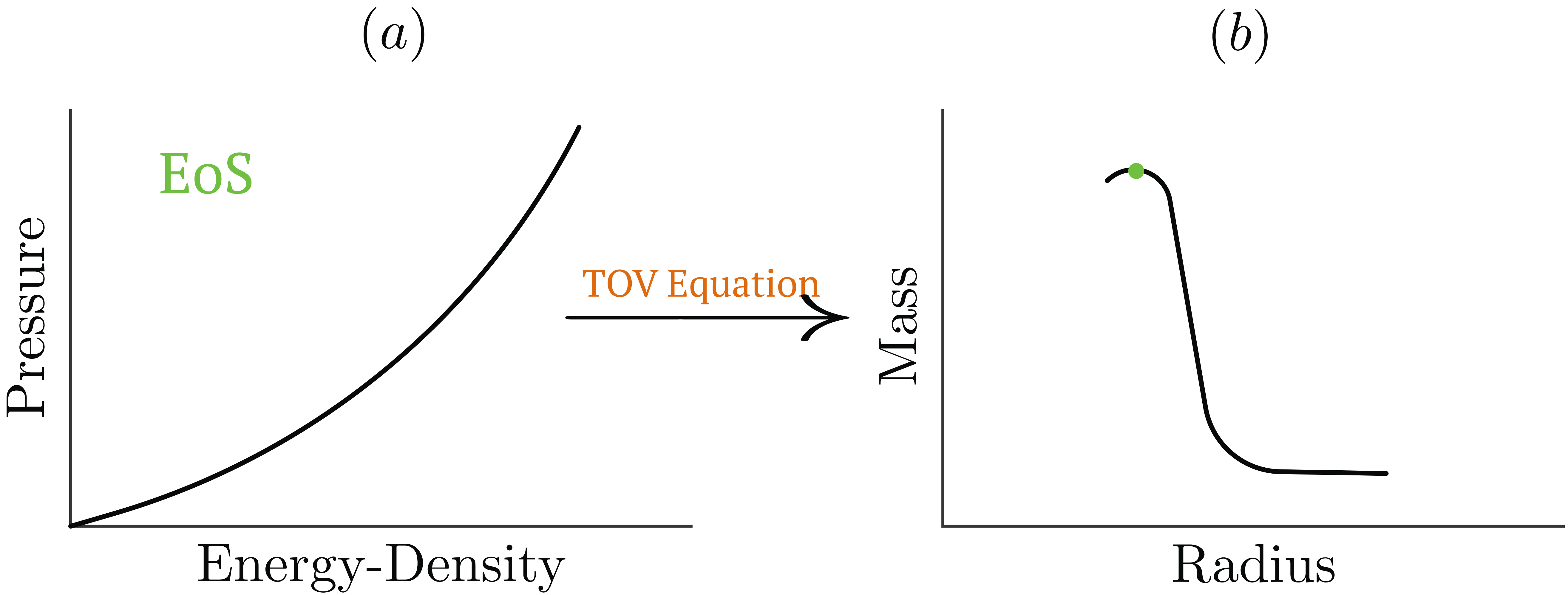

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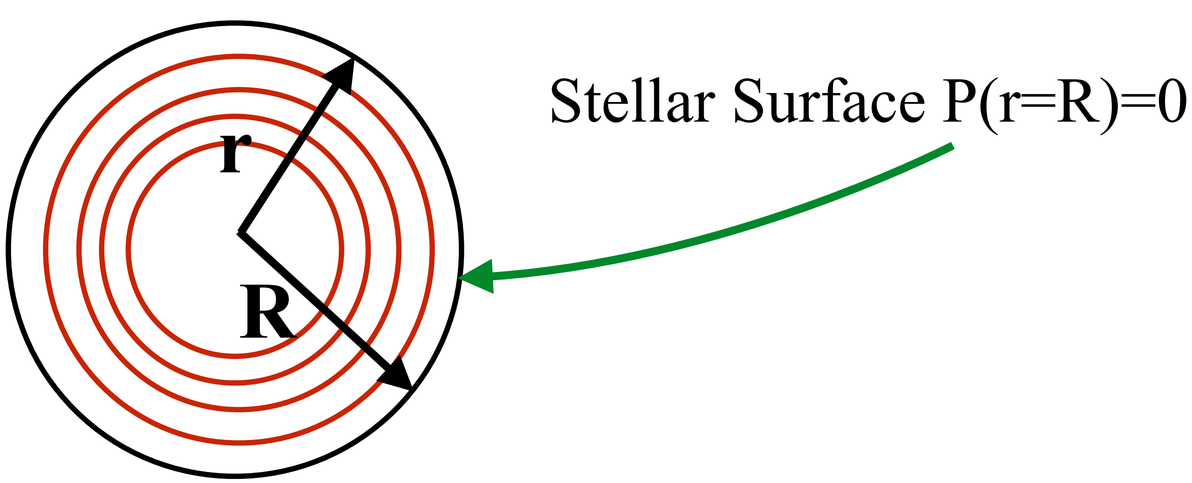

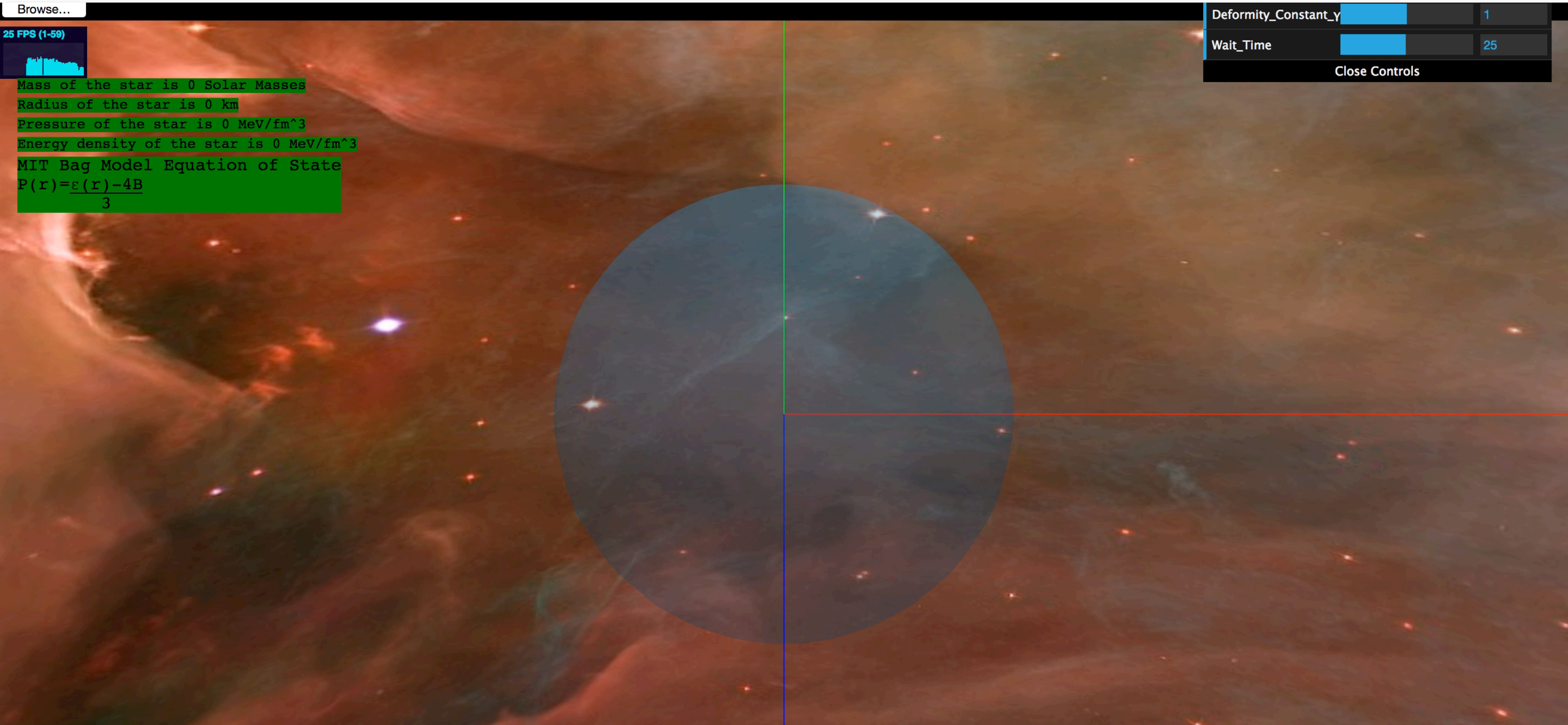

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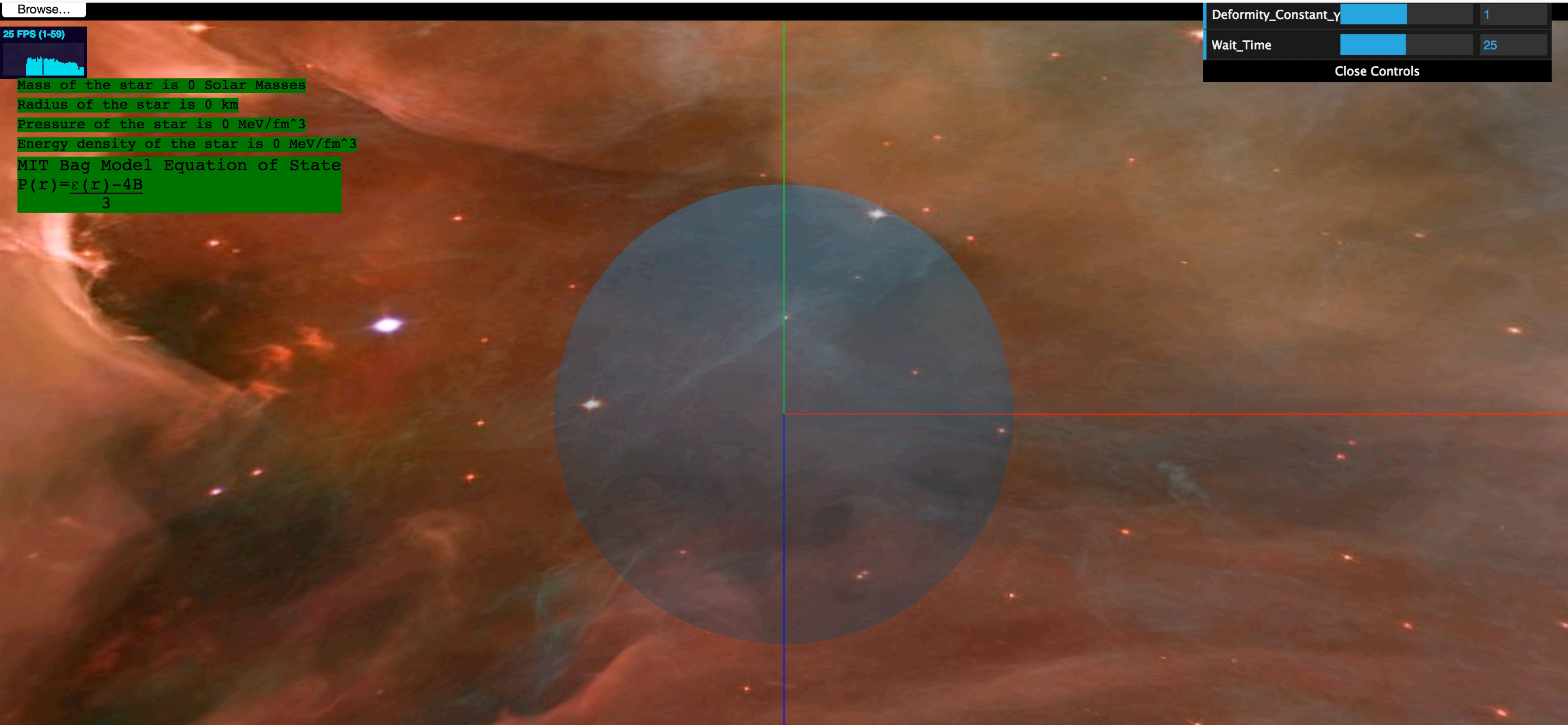

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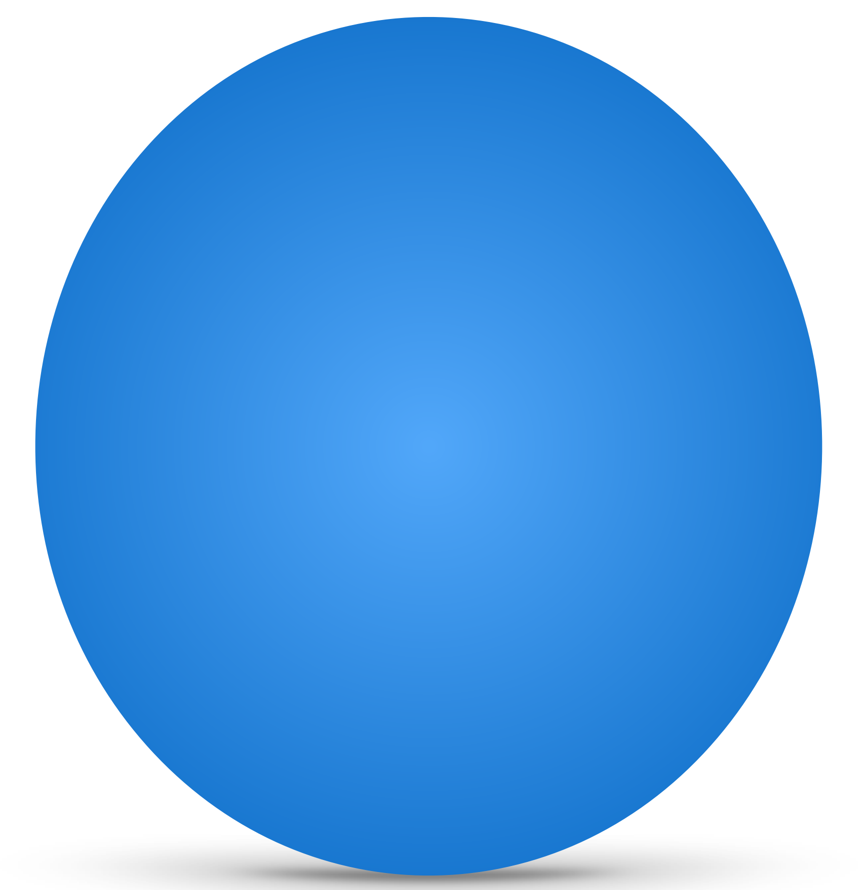

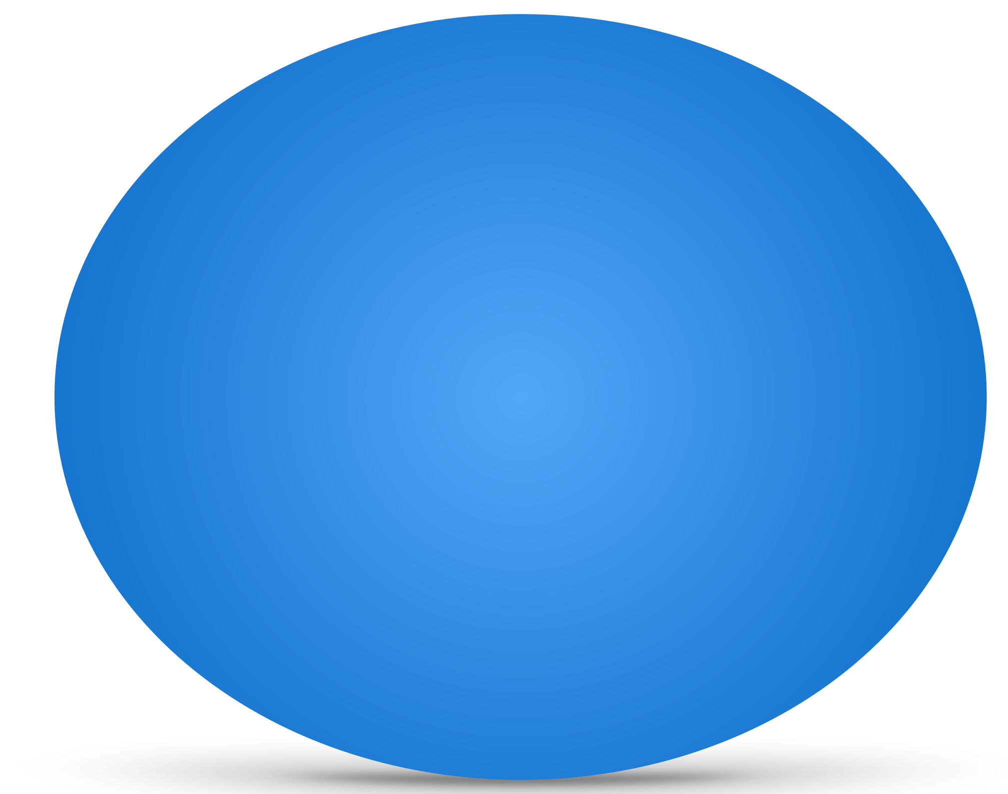

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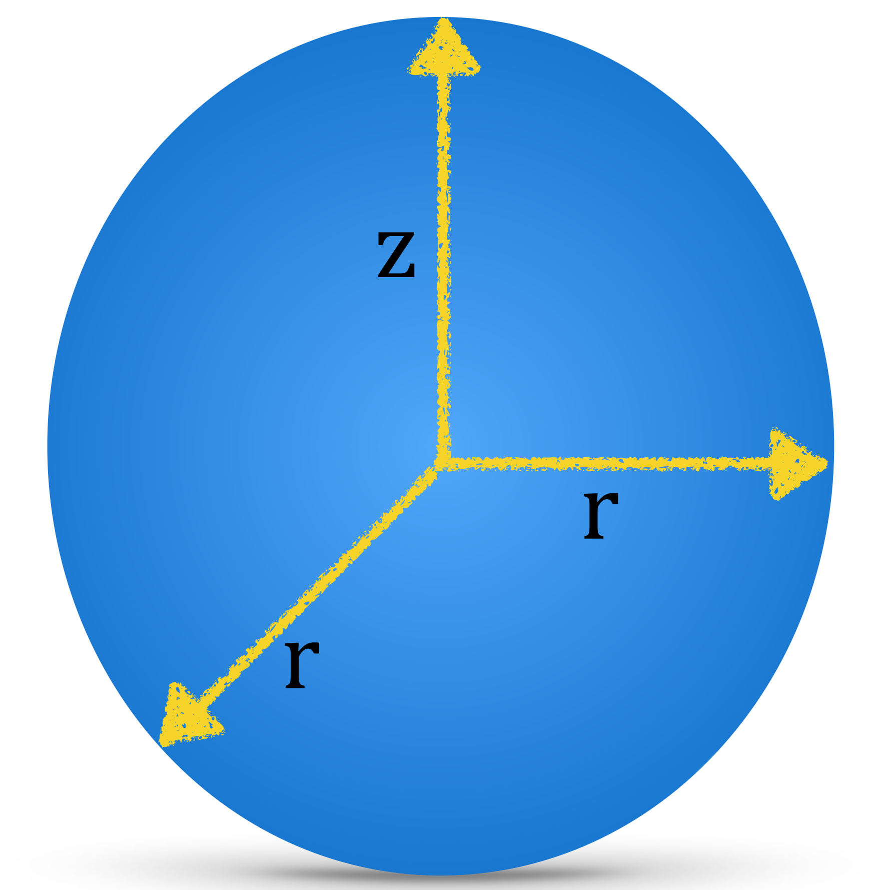

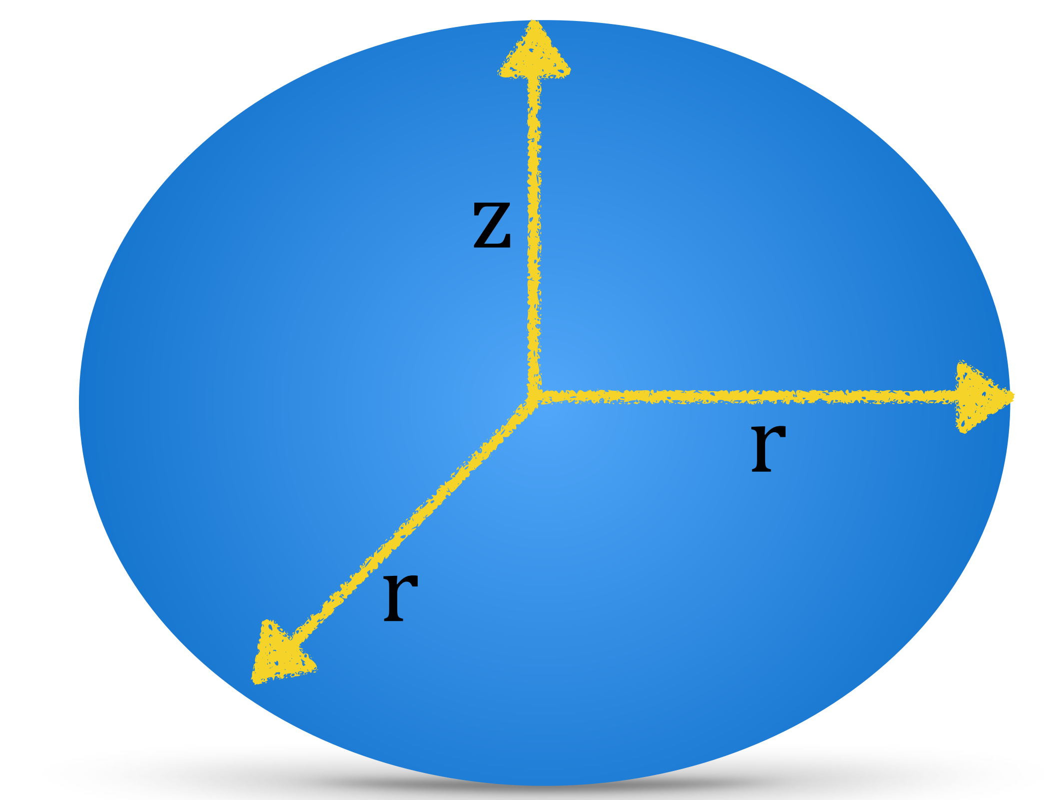

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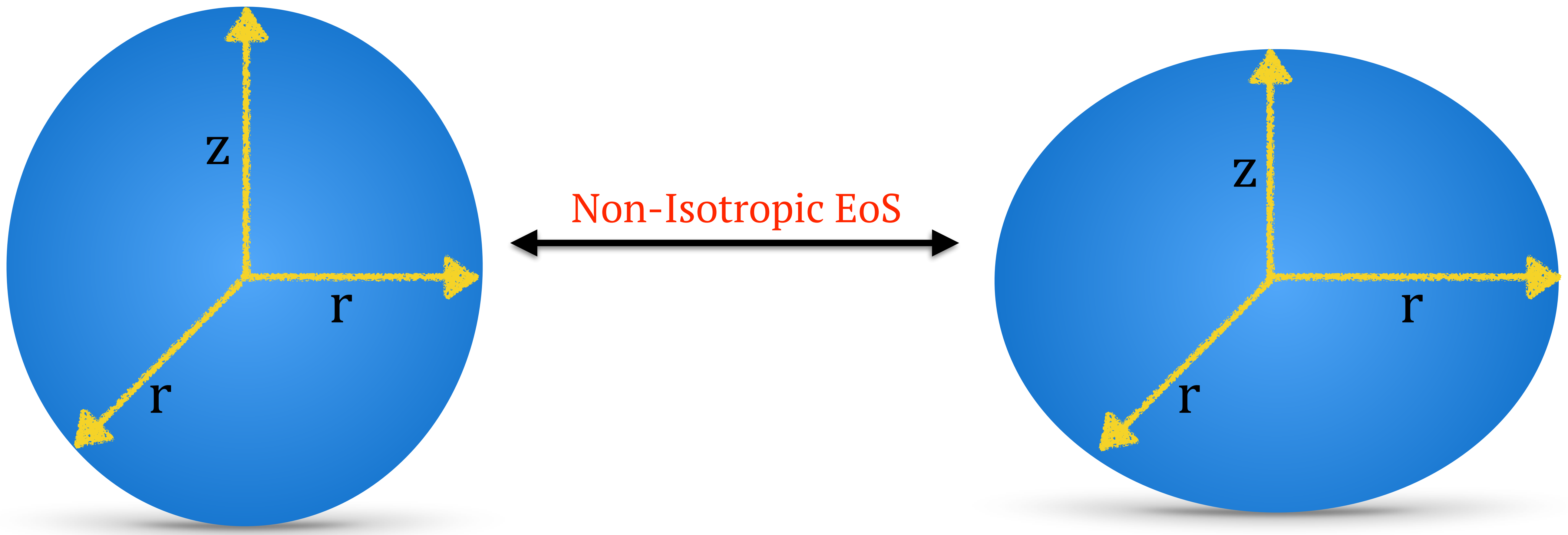

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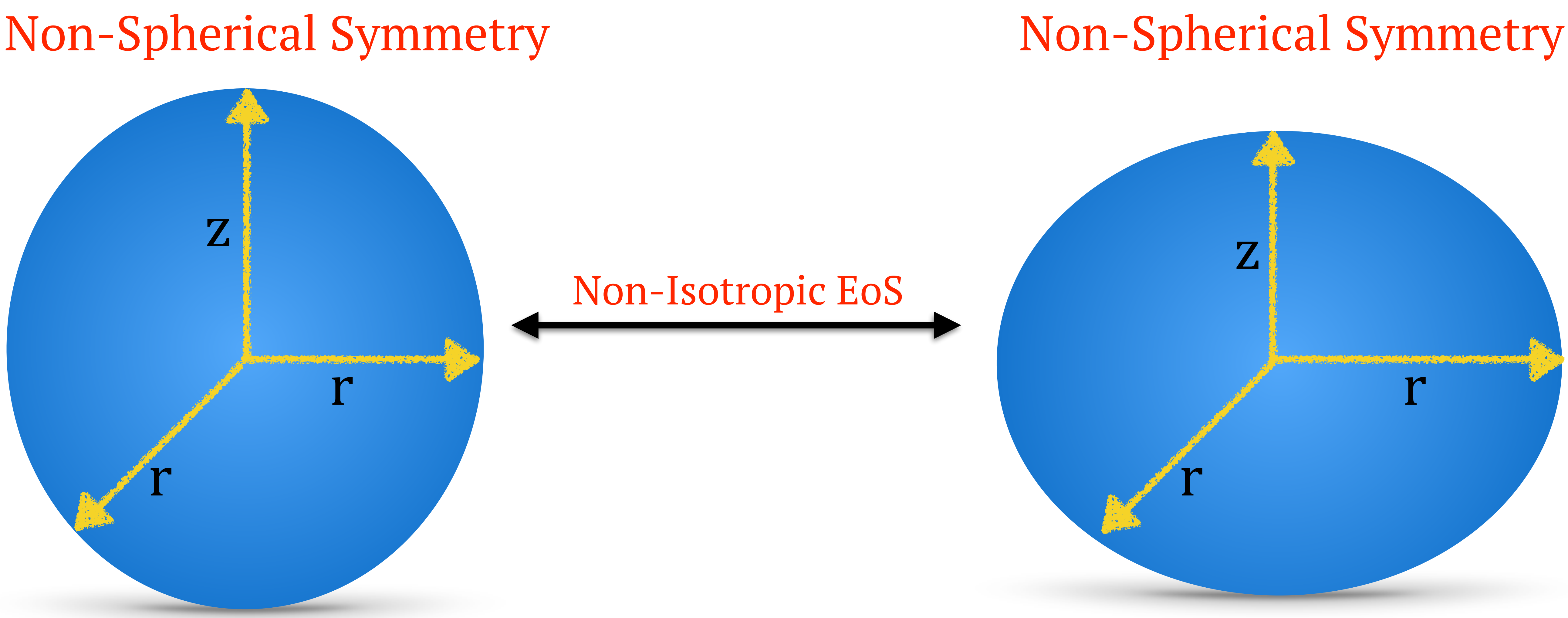

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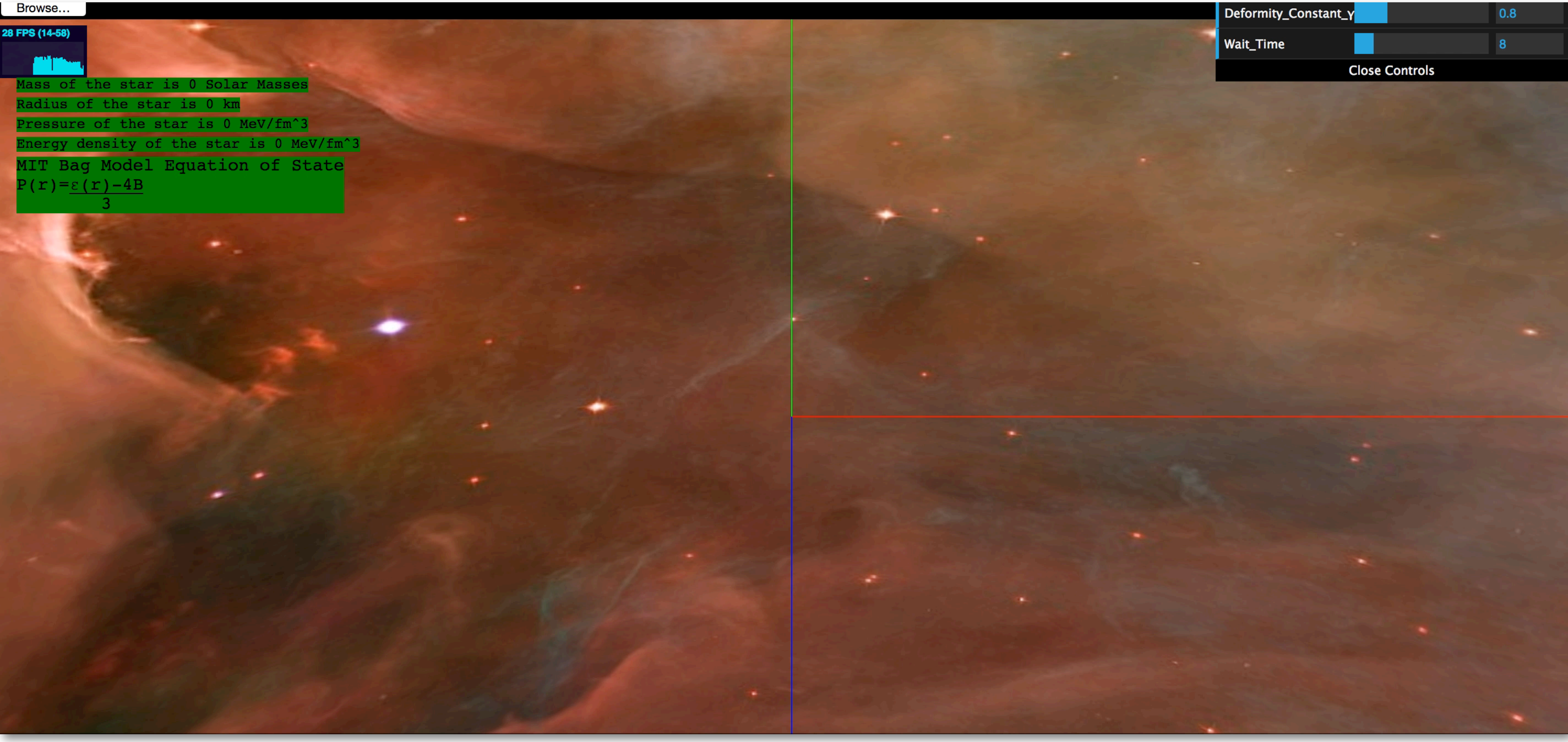

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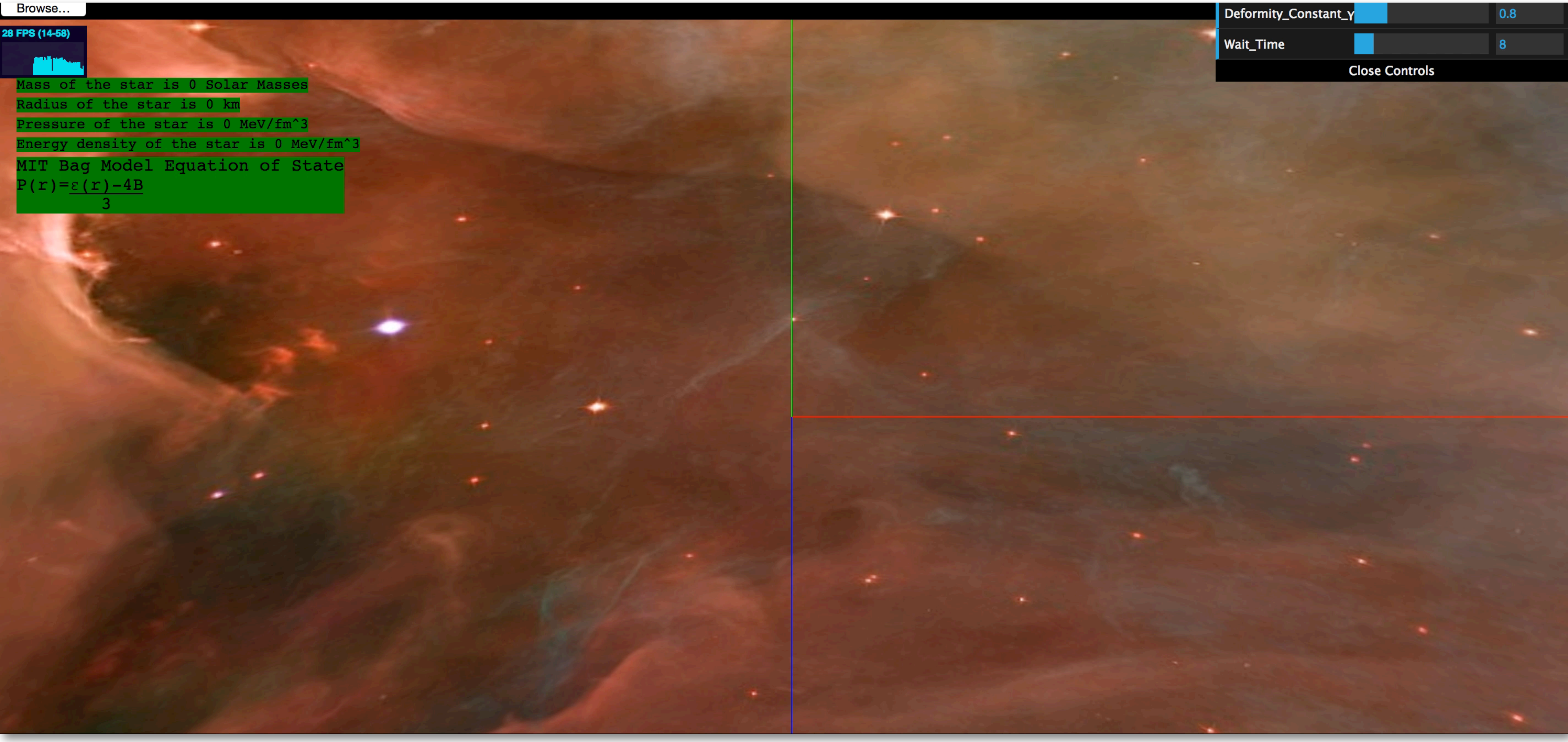

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PHYSICAL REVIEW C **82**, 065802 (2010)

Equation of state of a dense and magnetized fermion system

Efrain J. Ferrer, Vivian de la Incera, Jason P. Keith, Israel Portillo, and Paul L. Springsteen
Department of Physics, University of Texas at El Paso, El Paso, Texas 79968, USA
(Received 4 October 2010; published 10 December 2010)

The equation of state of a system of fermions in a uniform magnetic field is obtained in terms of the thermodynamic quantities of the theory by using functional methods. It is shown that the breaking of the O(3) rotational symmetry by the magnetic field results in a pressure anisotropy, which leads to the distinction between longitudinal- and transverse-to-the-field pressures. A criterion to find the threshold field at which the asymmetric regime becomes significant is discussed. This threshold magnetic field is shown to be the same as the one required for the pure field contribution to the energy and pressures to be of the same order as the matter contribution. A graphical representation of the field-dependent anisotropic equation of state of the fermion system is given. Estimates of the upper limit for the inner magnetic field in self-bound stars, as well as in gravitationally bound stars with inhomogeneous distributions of mass and magnetic fields, are also found.

DOI: 10.1103/PhysRevC.82.065802 PACS number(s): 21.65.Mn, 21.65.Qr, 26.60.Kp, 97.60.Jd

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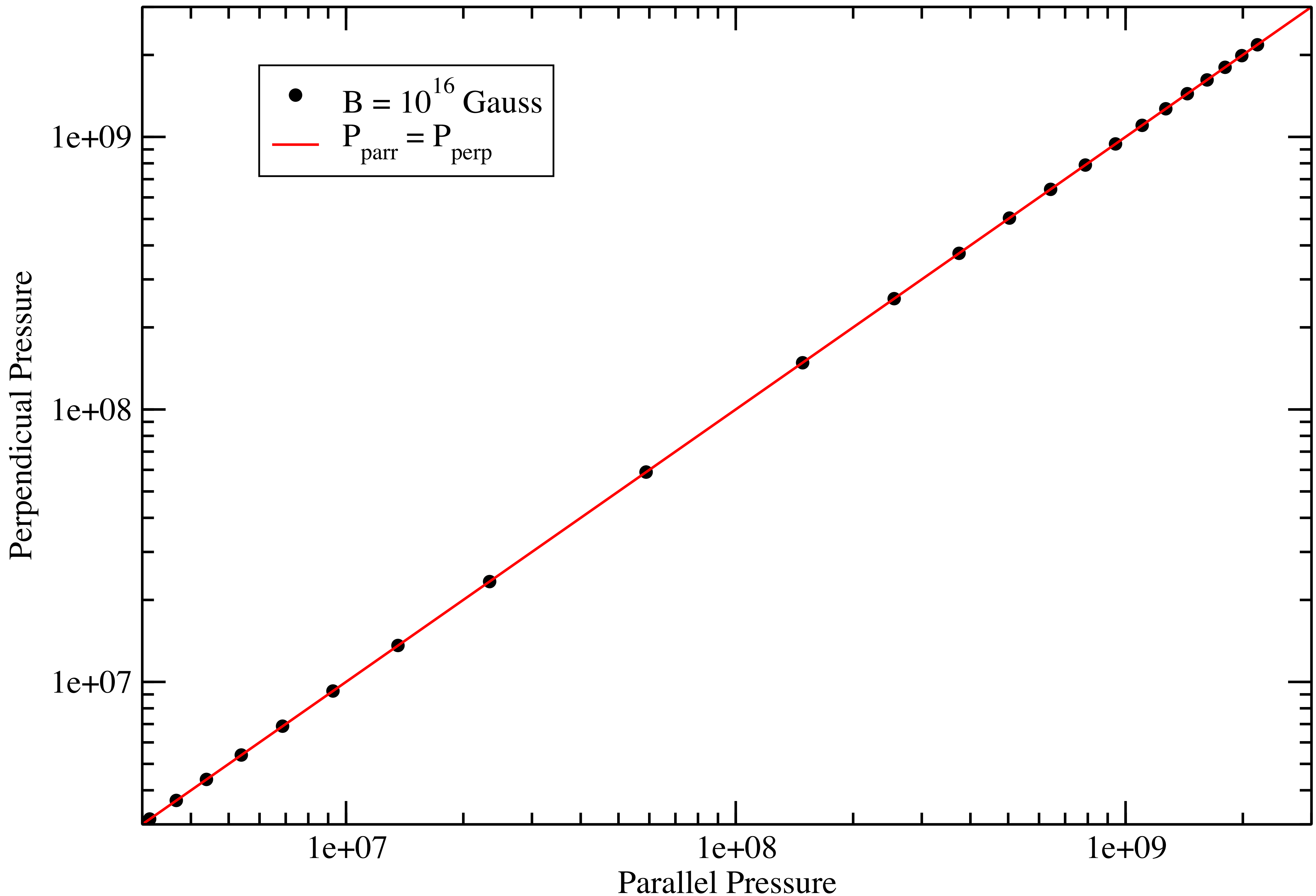

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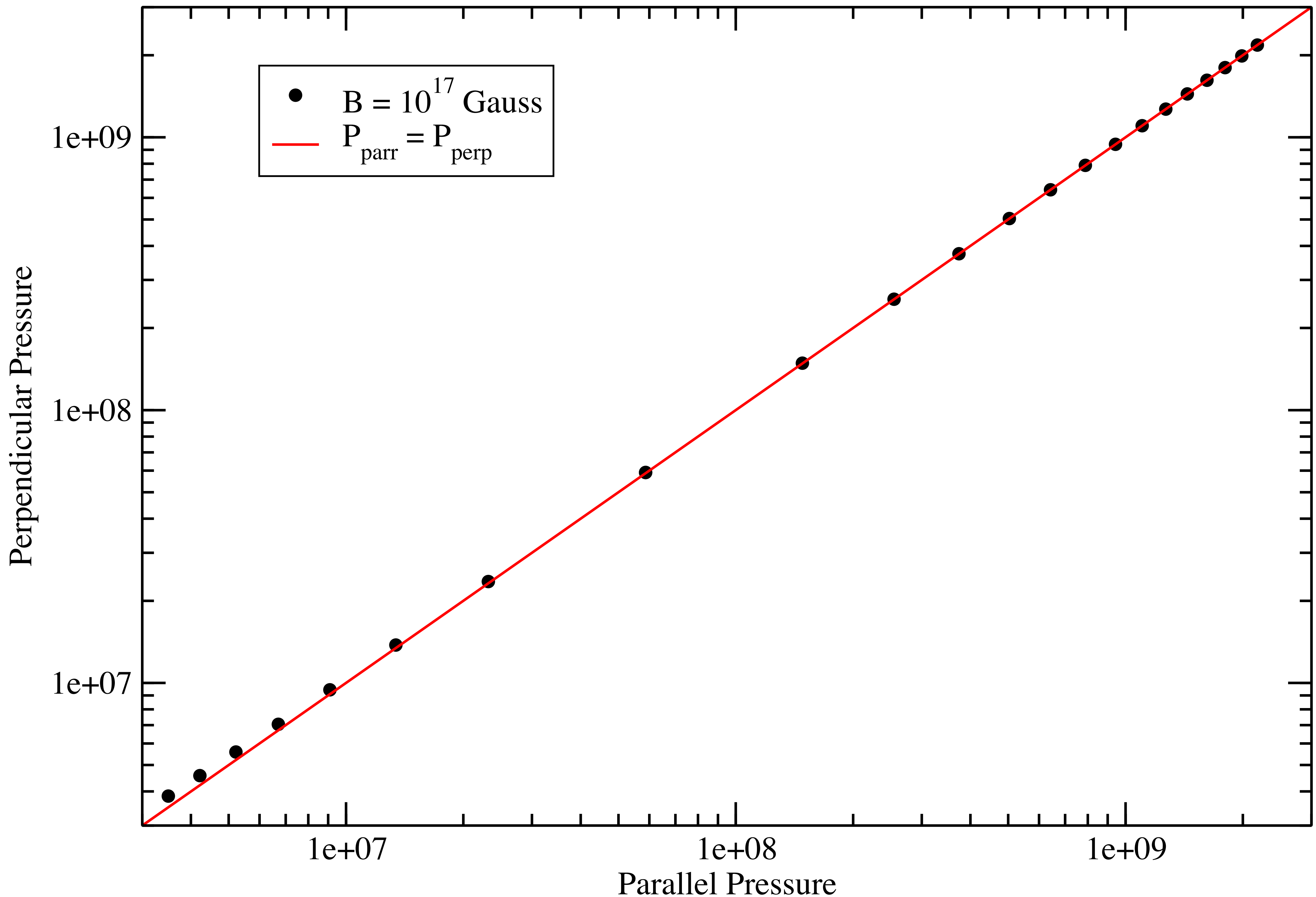

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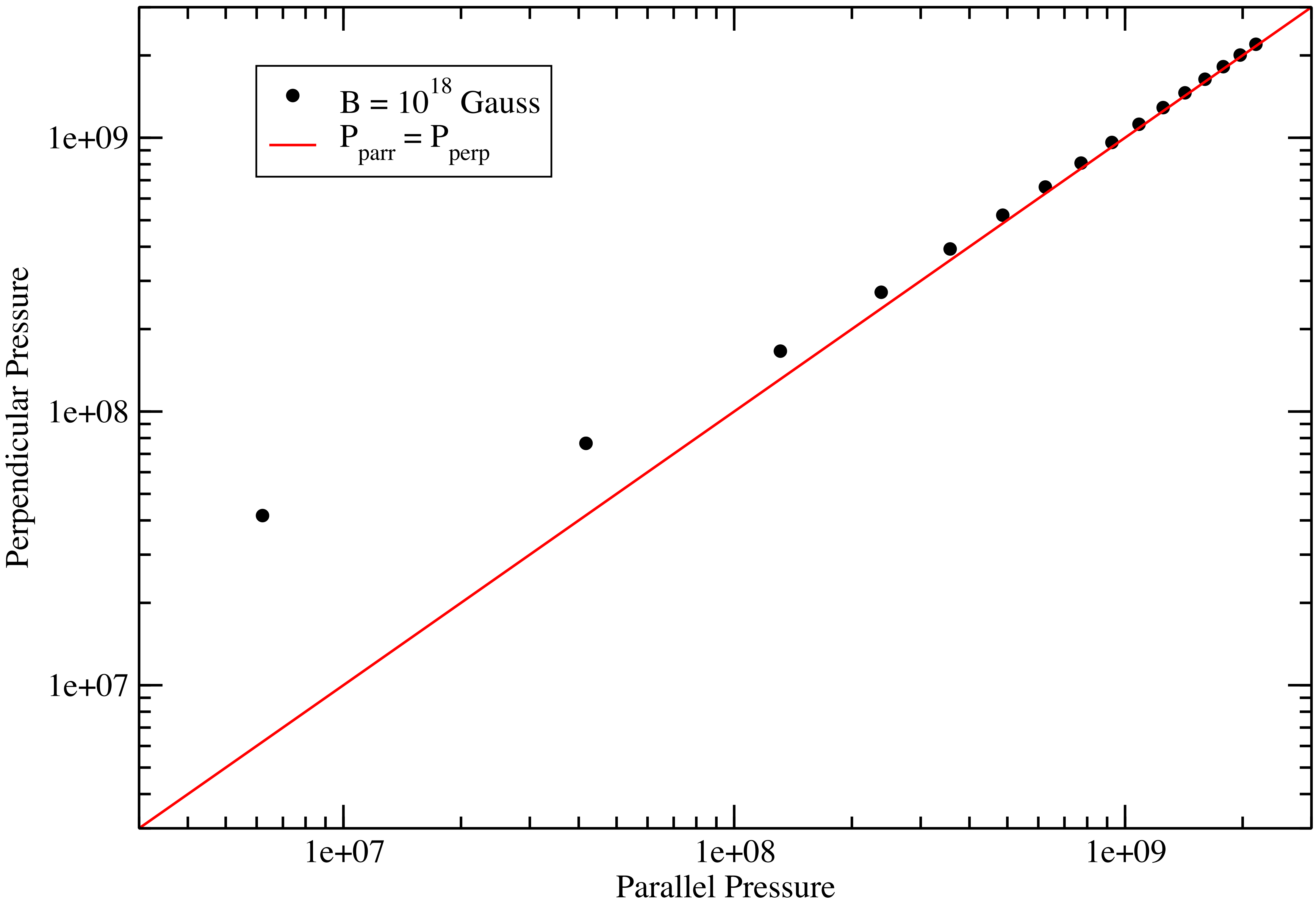

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- Goal is to obtain TOV-like equation(s) that will let us calculate stellar properties such as **masses** and **radii**.

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(Received 4 October 2010; published 10 December 2010)

The equation of state of a system of fermions in a uniform magnetic field is obtained in terms of the thermodynamic quantities of the theory by using functional methods. It is shown that the breaking of the O(3) rotational symmetry by the magnetic field results in a pressure anisotropy, which leads to the distinction between longitudinal- and transverse-to-the-field pressures. A criterion to find the threshold field at which the asymmetric regime becomes significant is discussed. This threshold magnetic field is shown to be the same as the one required for the pure field contribution to the energy and pressures to be of the same order as the matter contribution. A graphical representation of the field-dependent anisotropic equation of state of the fermion system is given. Estimates of the upper limit for the inner magnetic field in self-bound stars, as well as in gravitationally bound stars with inhomogeneous distributions of mass and magnetic fields, are also found.

DOI: 10.1103/PhysRevC.82.065802 PACS number(s): 21.65.Mn, 21.65.Qr, 26.60.Kp, 97.60.Jd

Motivation

- Hence, we will need derive stellar structure equations that will utilize this type of EoS model.
- Goal is to obtain TOV-like equation(s) that will let us calculate stellar properties such as **masses** and **radii**.
- Examine if deformation makes any significant changes to mass.

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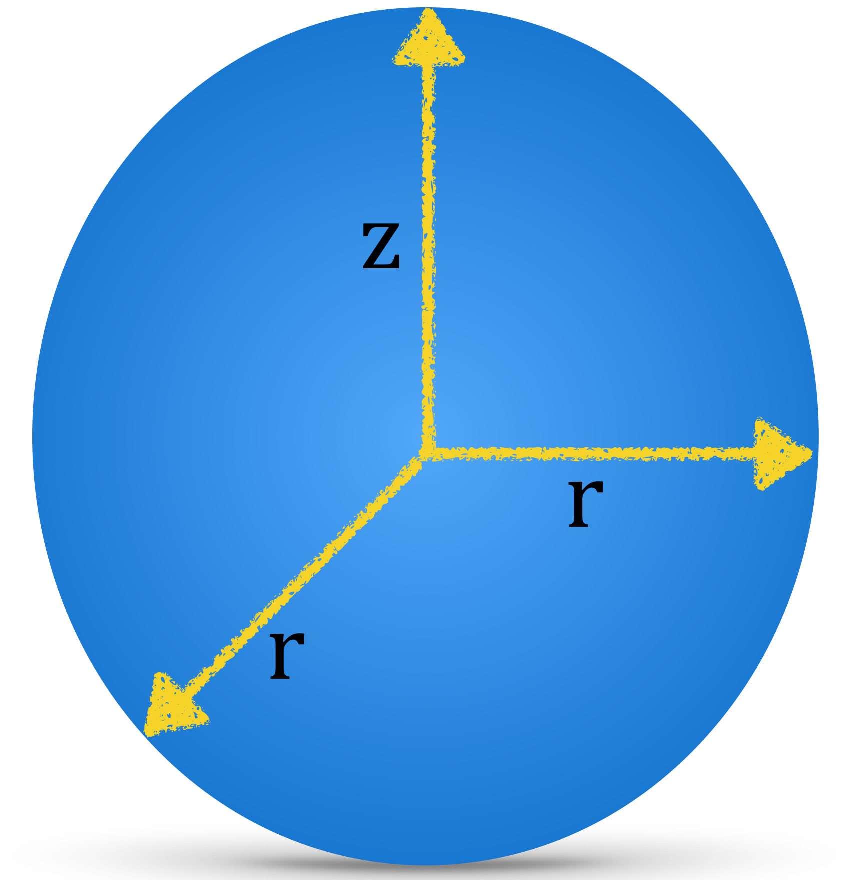
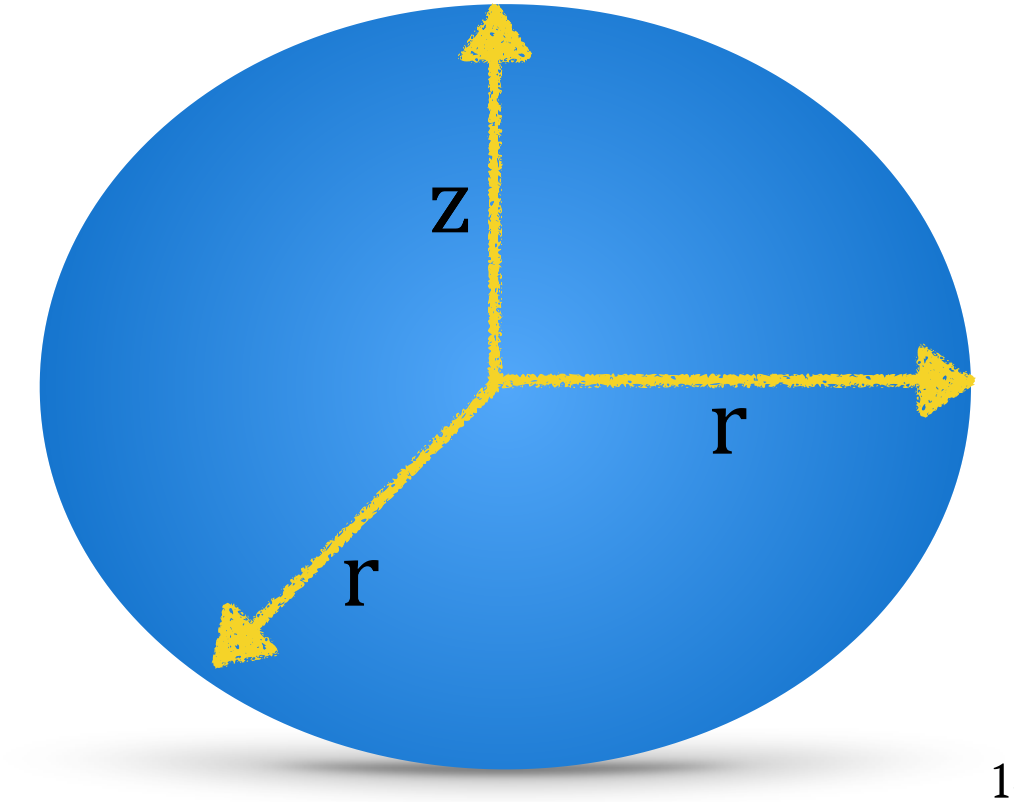
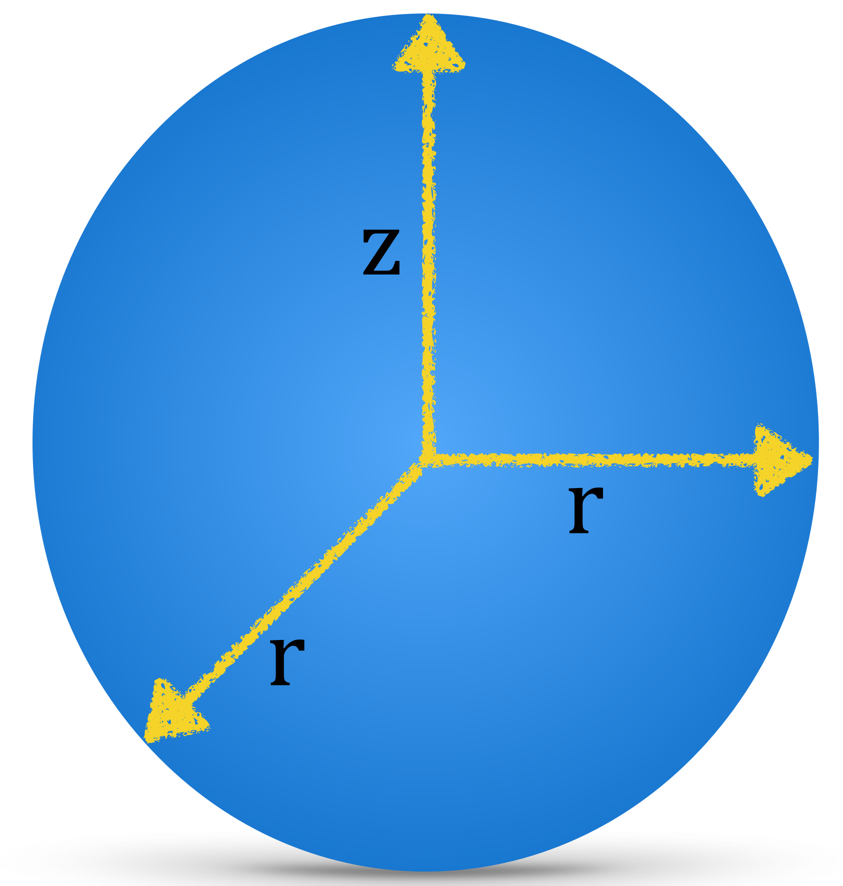
Consequences of Deformation

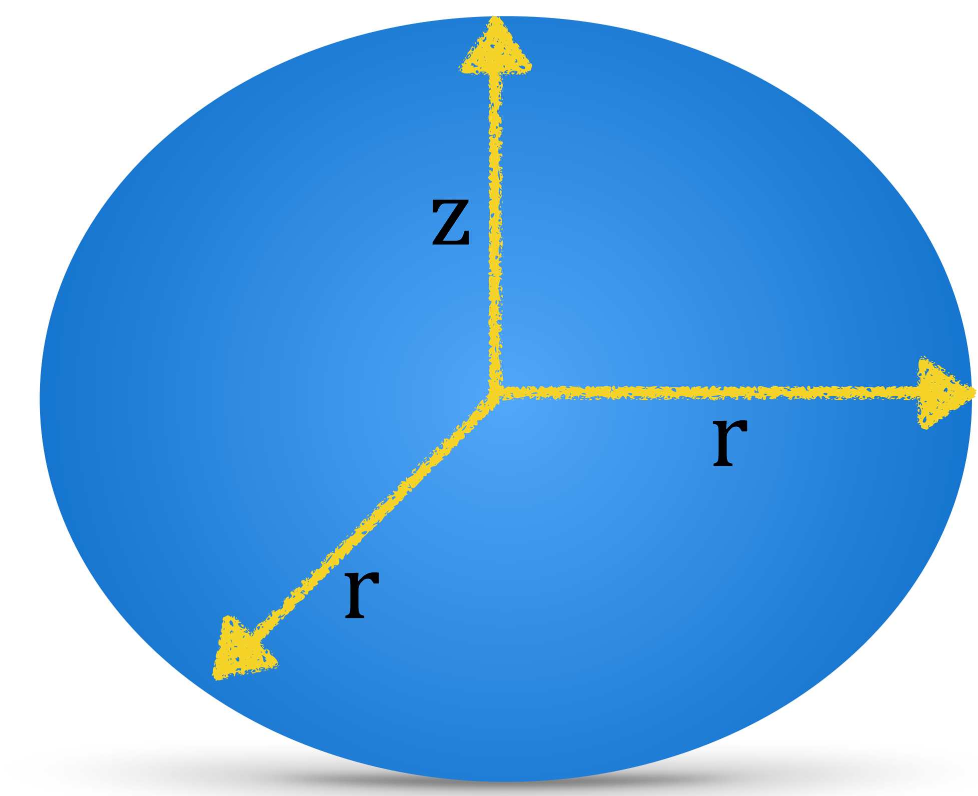




Consequences of Deformation

- The total gravitational mass of these deformed compact objects is expected to change...





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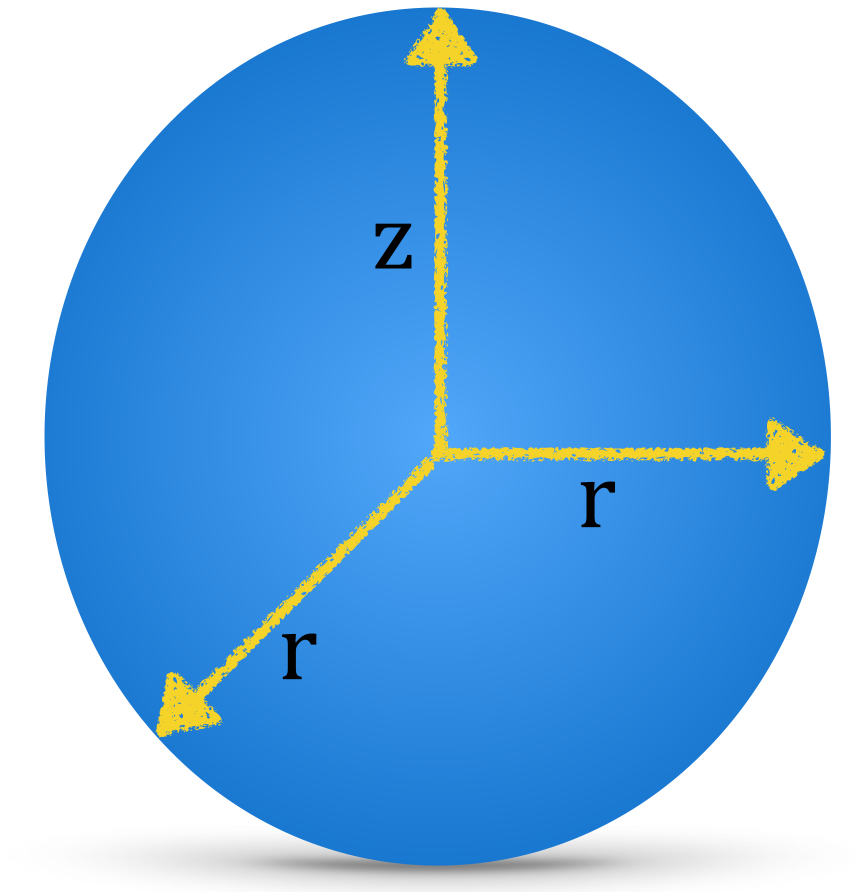
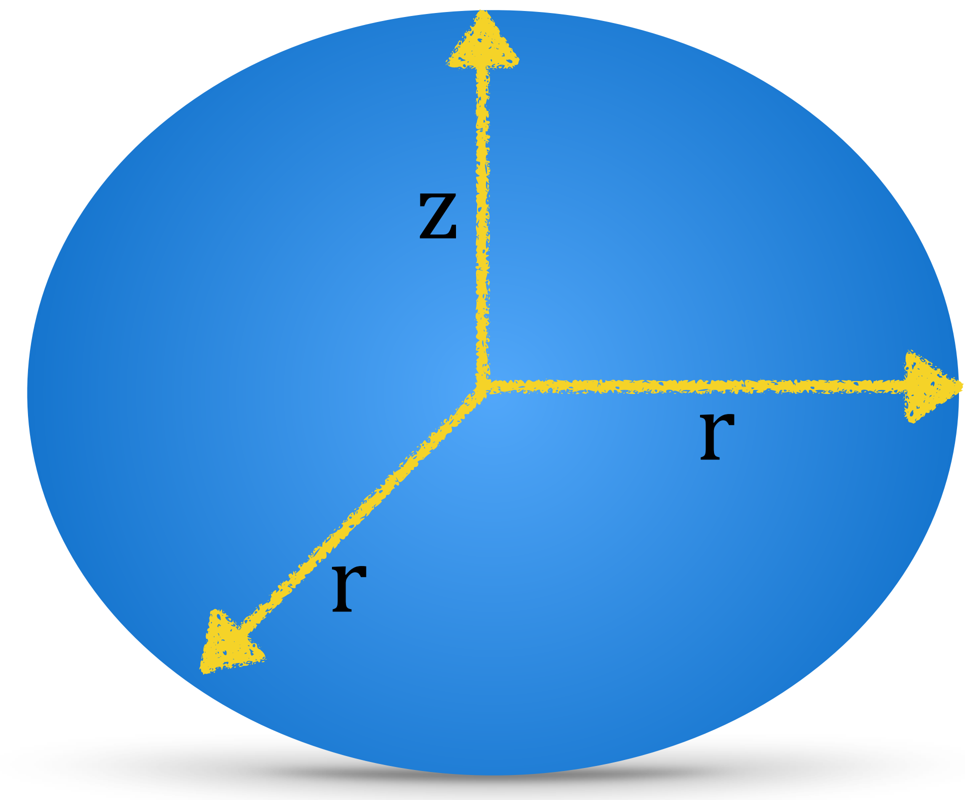
- Thus also leading to a different gravitational redshift z_{rs}…





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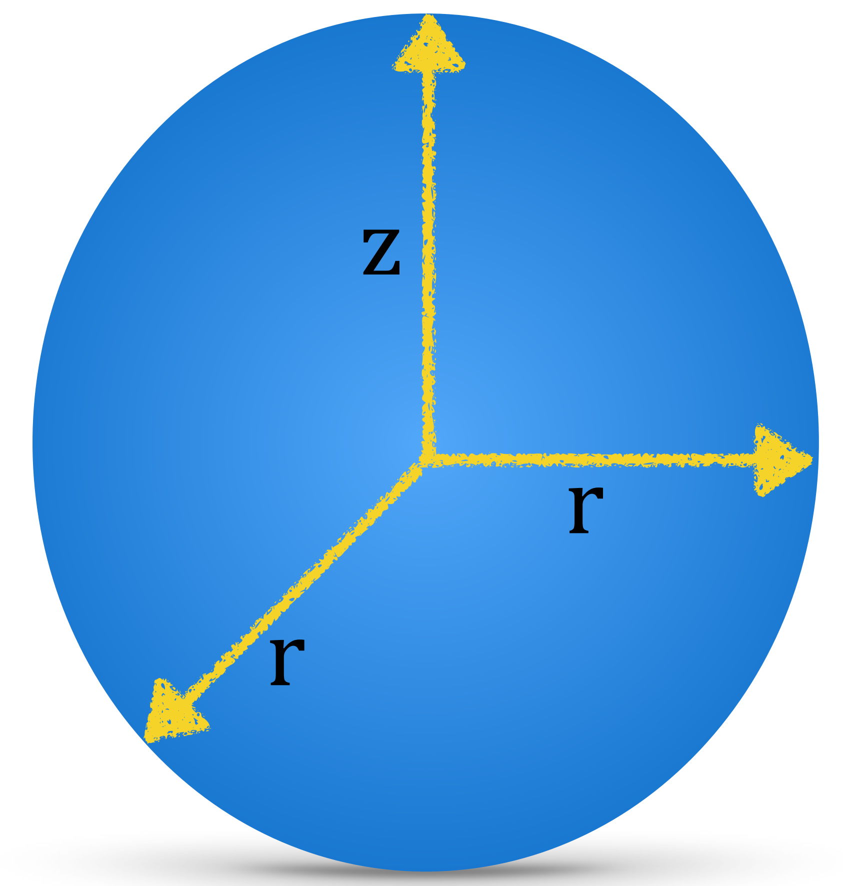
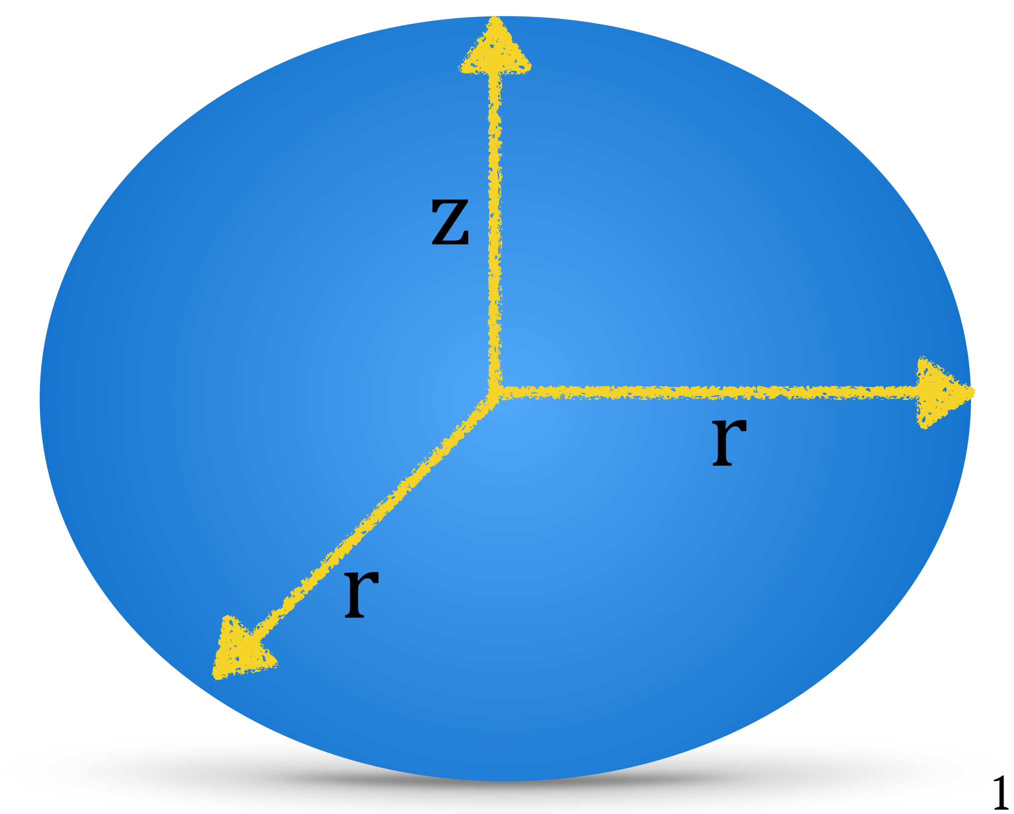
$$M_{\text{spherical}} \neq M_{\text{deformed}}$$
$$z_{\text{spherical}} \neq z_{\text{deformed}}$$





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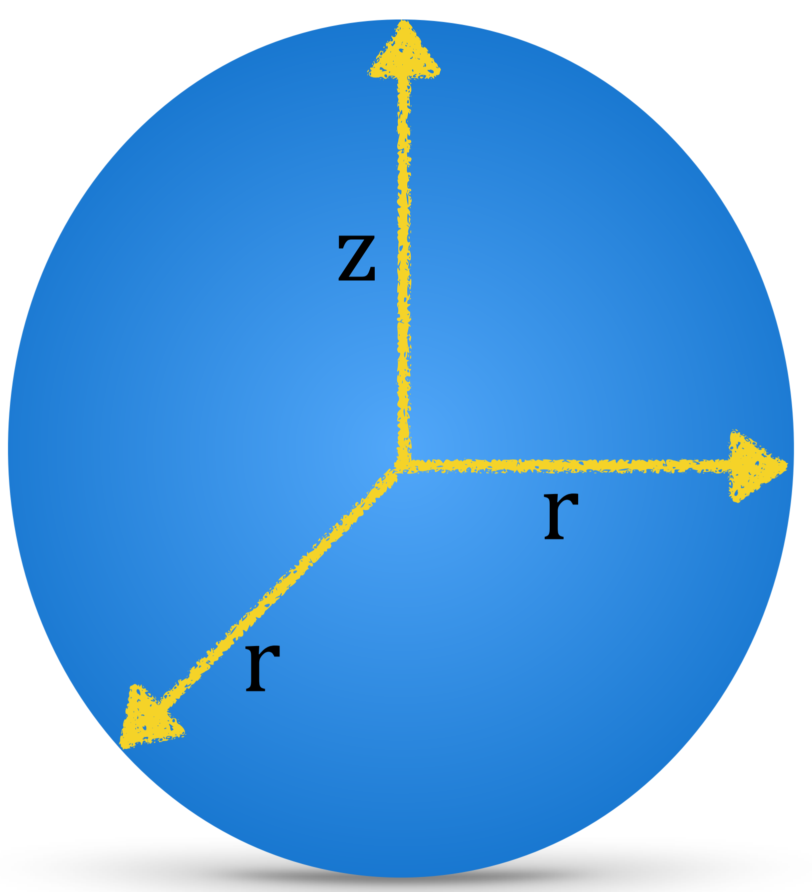
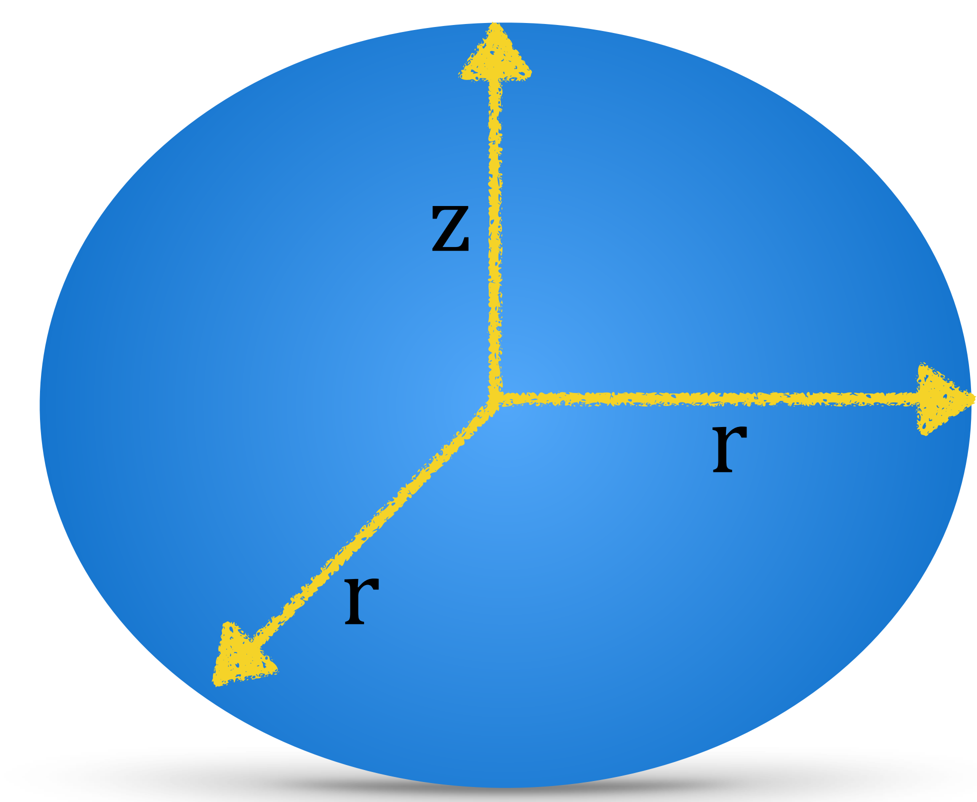
$$M_{\text{spherical}} \neq M_{\text{deformed}}$$
$$z_{\text{spherical}} \neq z_{\text{deformed}}$$

- Thus the internal properties (i.e. pressure) must approach zero both in the equatorial (**r**) and polar (**z**) directions...





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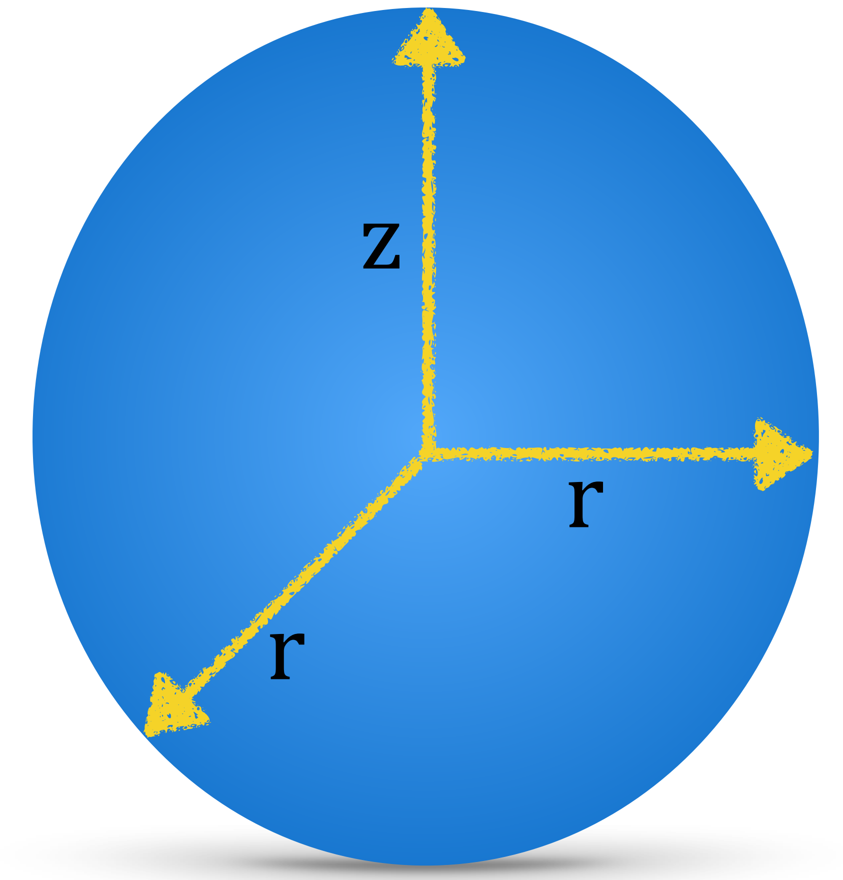
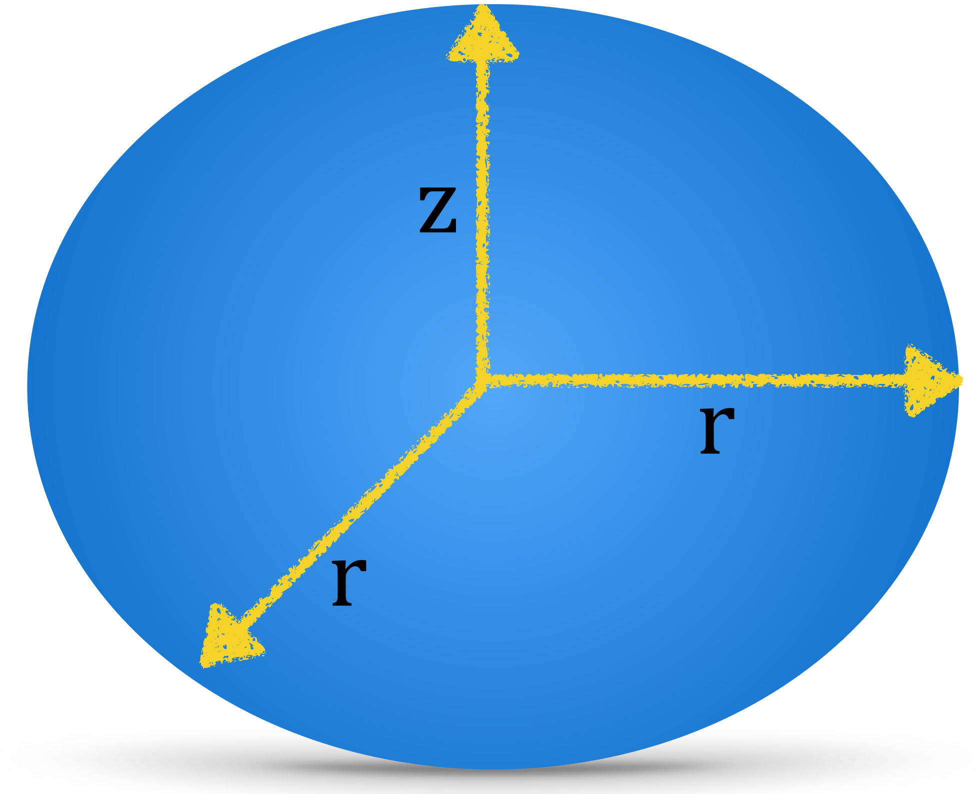
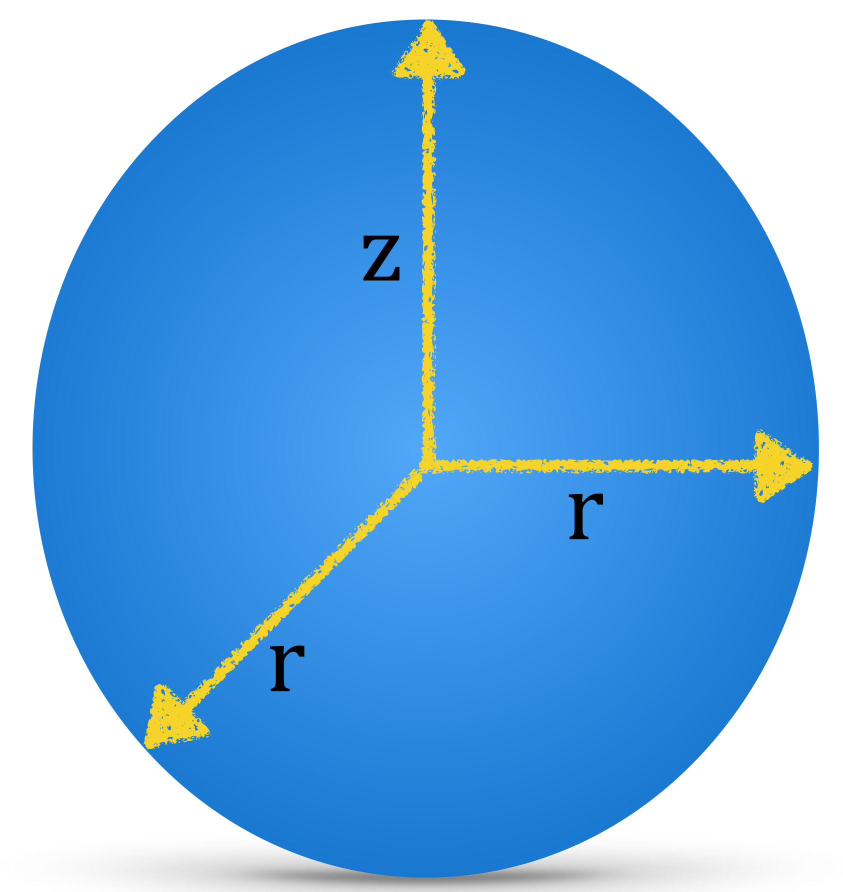
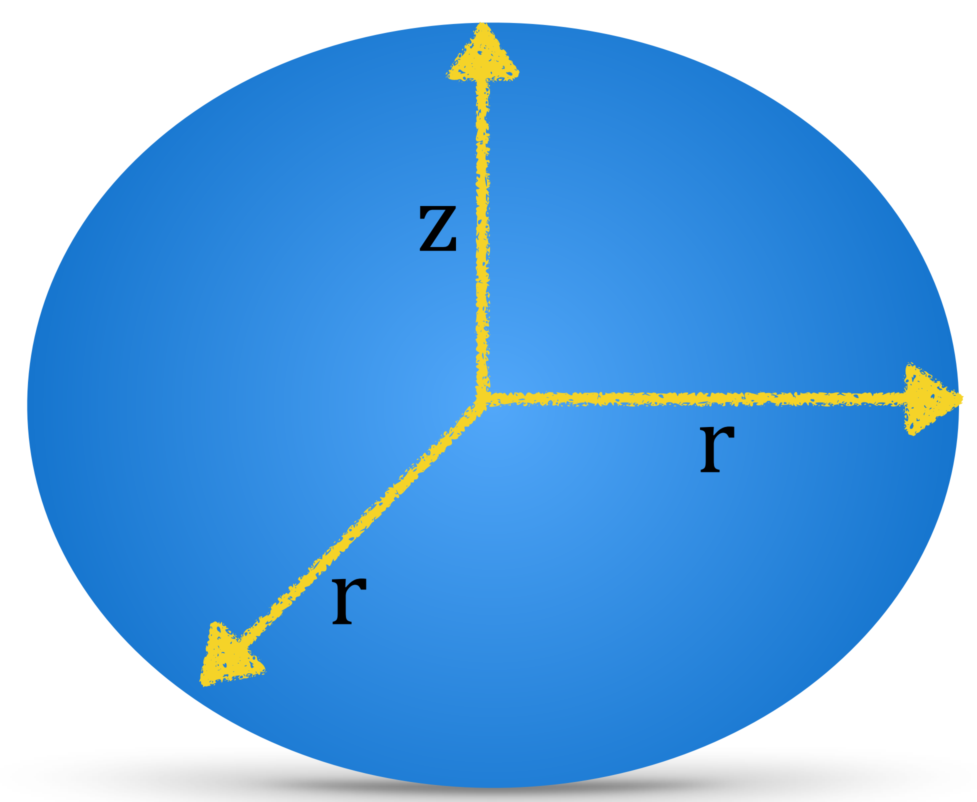




Consequences of Deformation

- Due to having distinct radial and polar directions, these oblate and prolate stars are expected to have a non-zero quadrupole moment as described by (*Herrera et al. 1999*):





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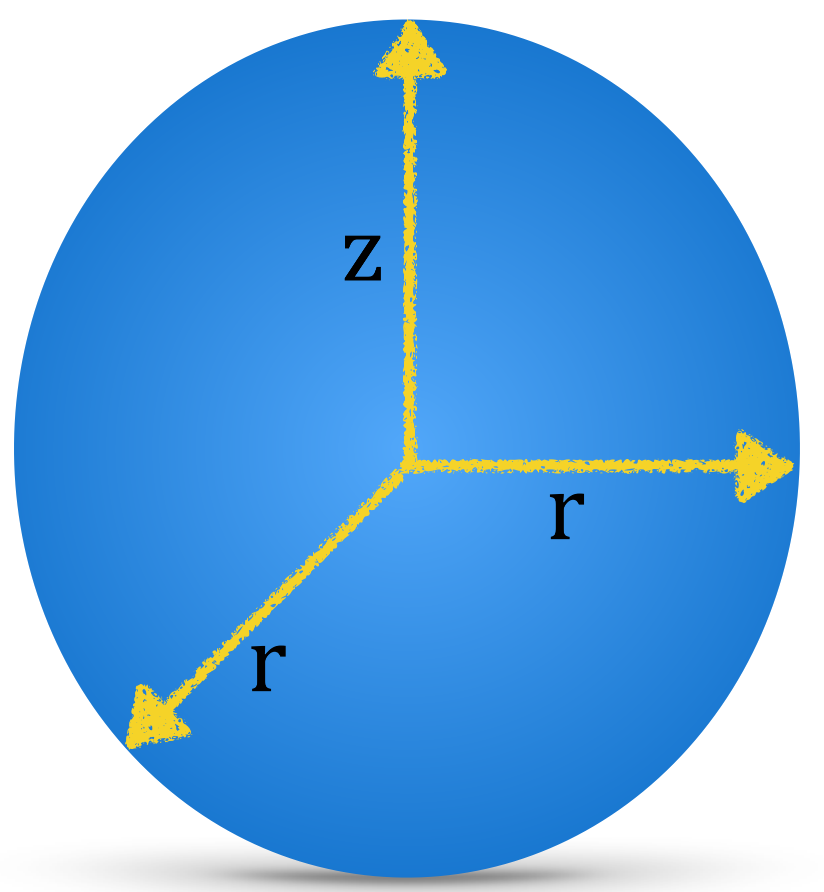

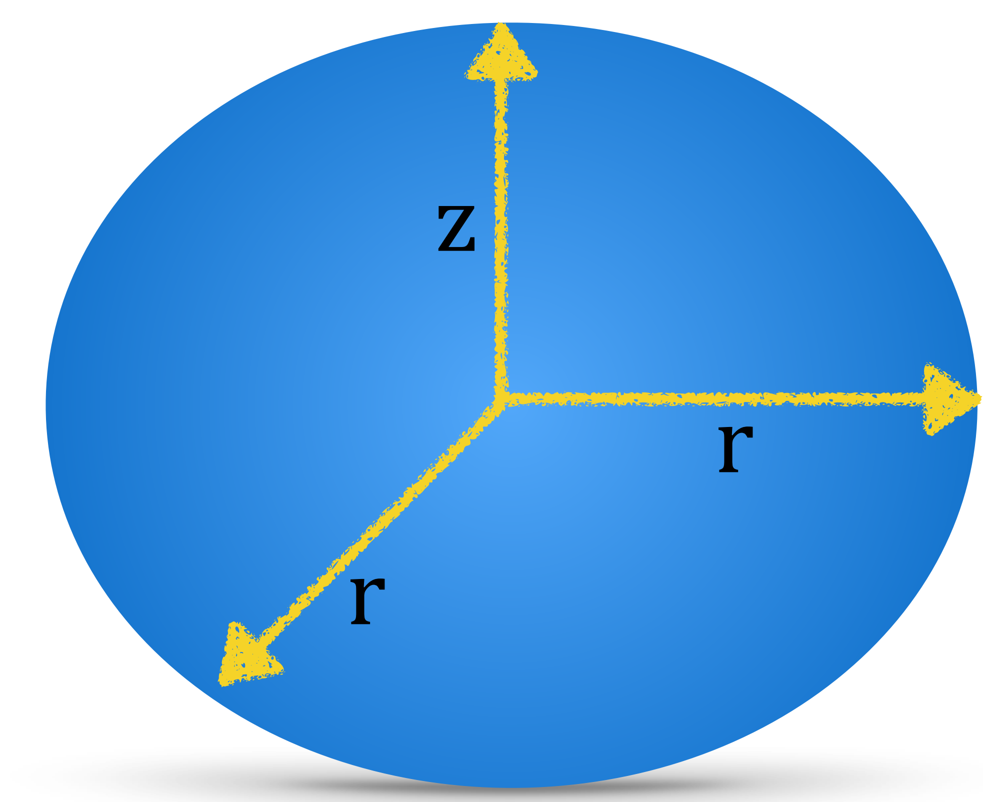

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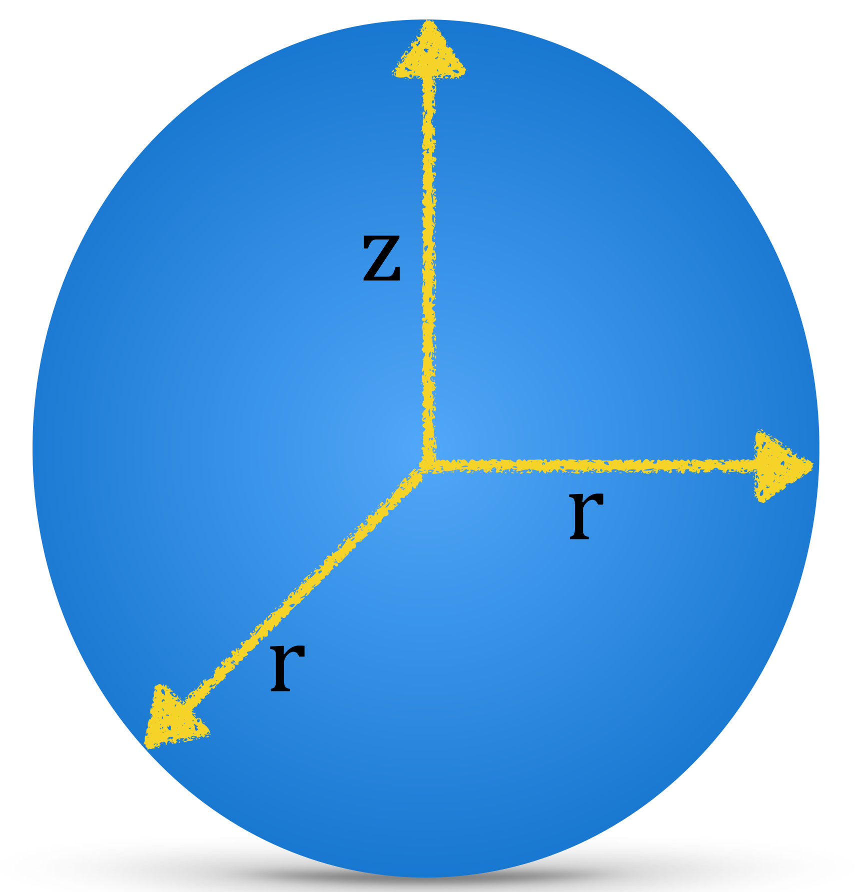
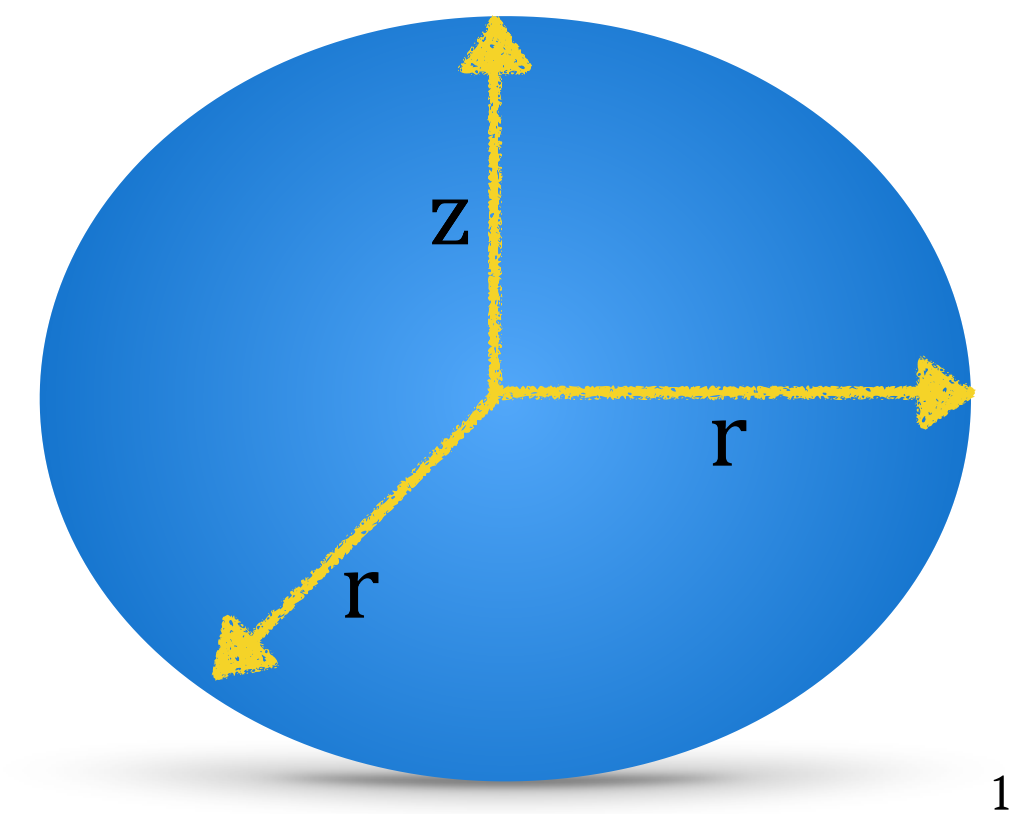
$M = \text{Total Mass}$

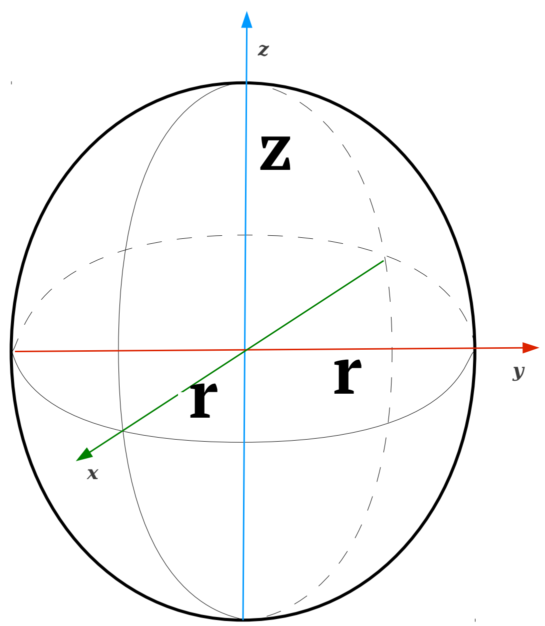
$\gamma = \text{Describes the degree of deformation}$





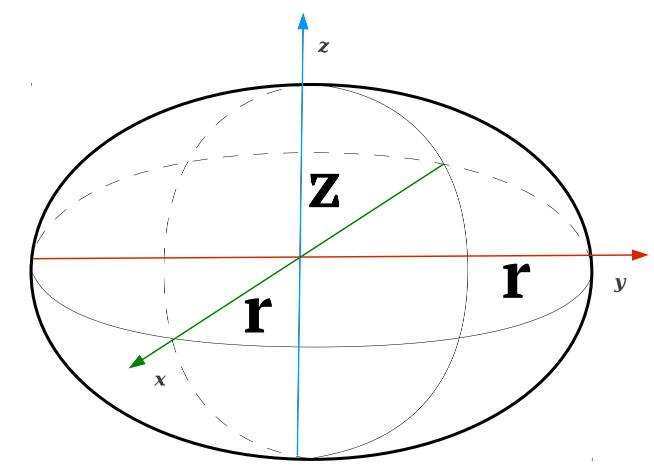
Non-Spherical Symmetry

Prolate spheroid with radii ***r*** & ***z***



Axially Symmetric

Oblate spheroid with radii ***r*** & ***z***



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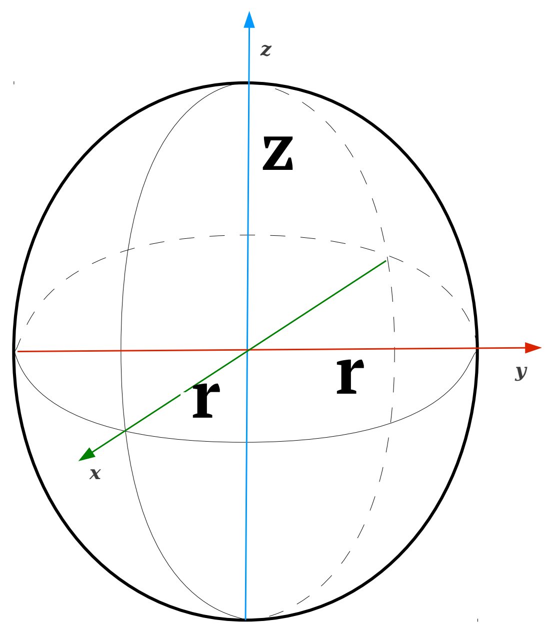

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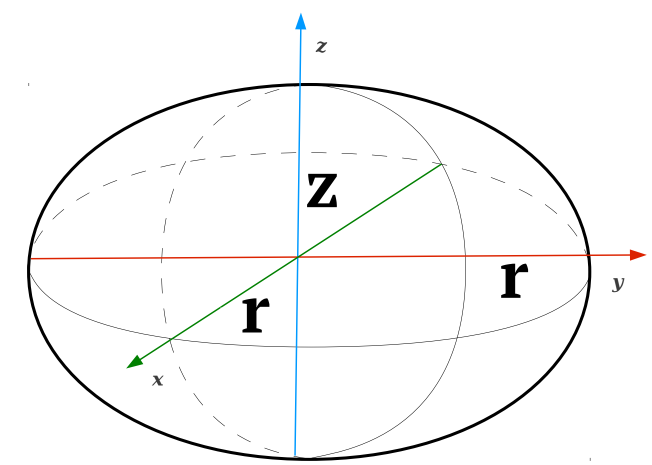

Weyl (*Weyl H. 1918*) Metric

$$ds^2 = \mathrm{e}^{2\Phi}\mathrm{dt}^2 - \mathrm{e}^{-2\Phi}\left[\mathrm{e}^{2\Lambda}\left(\mathrm{dr}^2 + \mathrm{dz}^2\right) + \mathrm{r}^2\mathrm{d}\phi^2\right]$$

Parameterization

Since we have ***distinct*** radial and polar directions—Need axially symmetric structure equations in this framework using General Relativity (Field Equations…)

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- Simplify this scenario by **parameterizing** the z-component.

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- Simplify this scenario by **parameterizing** the z-component.
  - Will allow us to use an EoS in the **limiting case** of Isotropic energy-density.
  - Still maintain **deformation structure**.
- Calculate stellar properties such as **mass** and **radii**.
- Investigate any changes from the standard spherical model.

Parameterization

We start with the Weyl (*Weyl H. 1918*) metric in non-spherical coordinates given by:

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Metric functions: $\Phi = \Phi(r, z),\ \Lambda = \Lambda(r, z)$

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where:

$$\mathrm{e}^{2\Phi} = \left[\frac{\mathrm{k}_1 + \mathrm{k}_2 - 2\mathrm{m}}{\mathrm{k}_1 + \mathrm{k}_2 + 2\mathrm{m}}\right]^{\gamma}, \;\; \mathrm{e}^{2\Lambda} = \left[\frac{(\mathrm{k}_1 + \mathrm{k}_2 + 2\mathrm{m})(\mathrm{k}_1 + \mathrm{k}_2 - 2\mathrm{m})}{4\mathrm{k}_1\mathrm{k}_2}\right]^{\gamma^2}$$

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and: $$k_1 = \sqrt{r^2 + (z-m)^2}, \quad k_2 = \sqrt{r^2 + (z+m)^2}$$

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and: $k_1 = \sqrt{r^2 + (z - m)^2} \ , \quad k_2 = \sqrt{r^2 + (z + m)^2}$

In the case for **γ =1**, we obtain the metric for a spherically symmetric object and will have the form (*Esposito & Witten 1975, Herrera et al. 1999*):

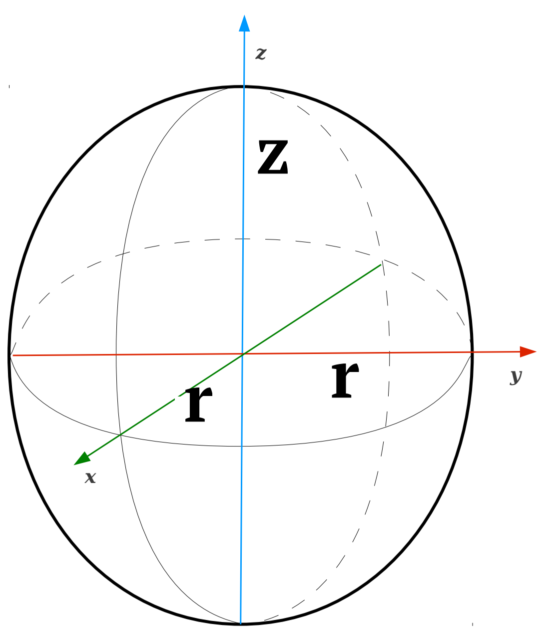
$$ds^2 = Adt^2 - A^{-1}\left[Bdr^2 + Cd\theta^2 + \left(r^2 - 2mr\right)\sin^2(\theta)d\phi^2\right]$$

Parameterization

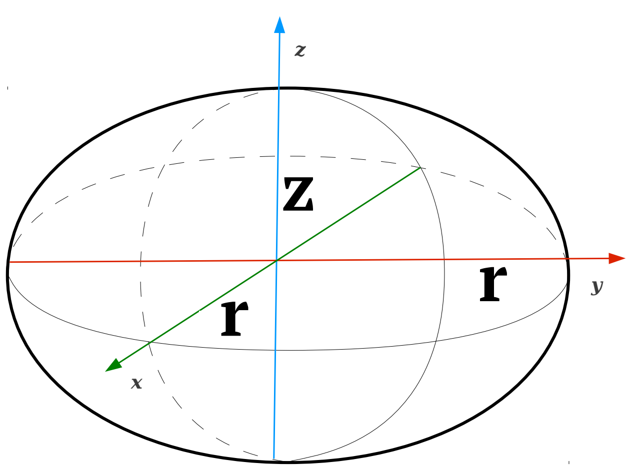
$$ds^2 = Adt^2 - A^{-1}\left[Bdr^2 + Cd\theta^2 + \left(r^2 - 2mr\right)\sin^2(\theta)d\phi^2\right]$$

Where the terms:

$$A = \left(1 - \frac{2m}{r}\right)^{\gamma} \ , \ B = \left(\frac{r^2 - 2mr}{r^2 - 2mr + m^2\sin^2(\theta)}\right)^{\gamma^2 - 1} \ , \ C = \frac{\left(r^2 - 2mr\right)^{\gamma^2}}{\left(r^2 - 2mr + m^2\sin^2(\theta)\right)^{\gamma^2 - 1}}$$



Prolate Spheroid



Oblate Spheroid

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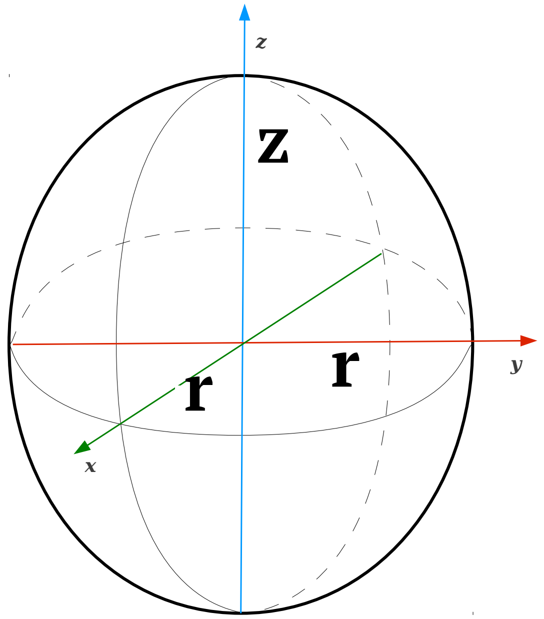
γ = 1



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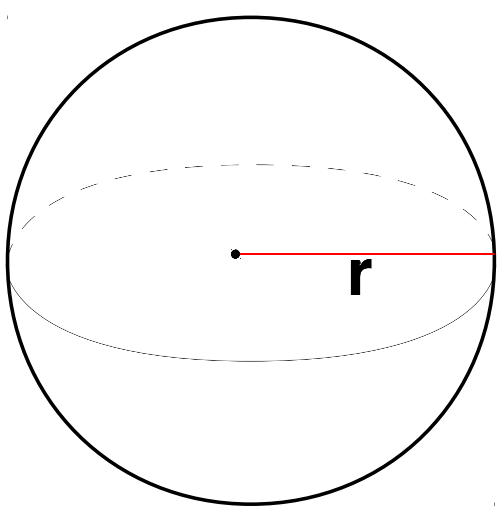
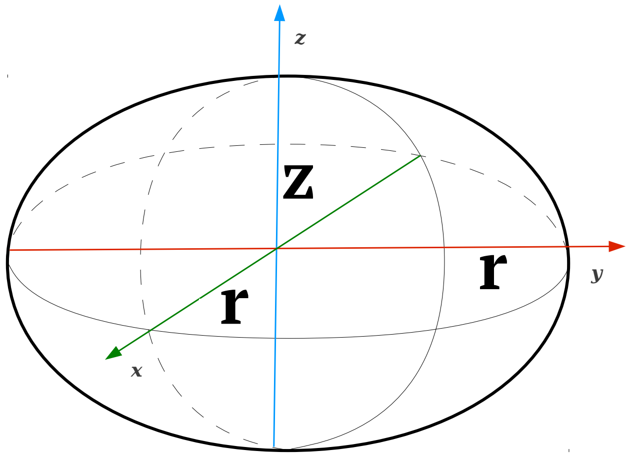


z=γr



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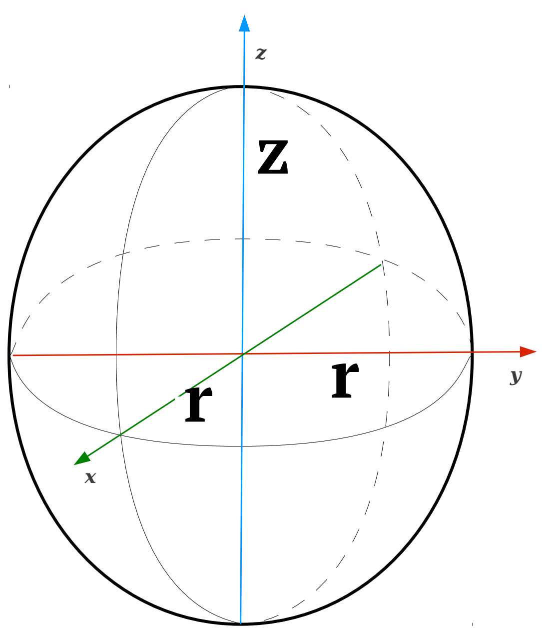
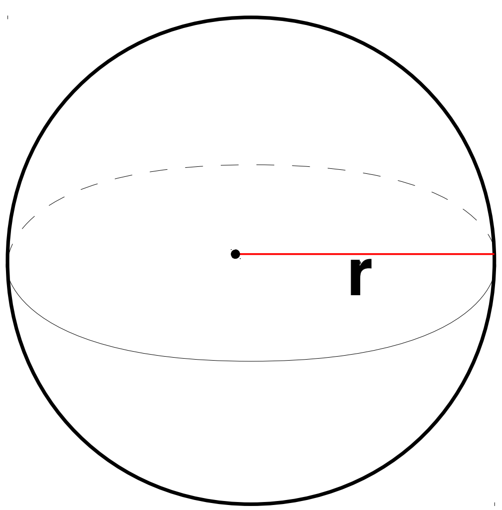
$\gamma > 1$



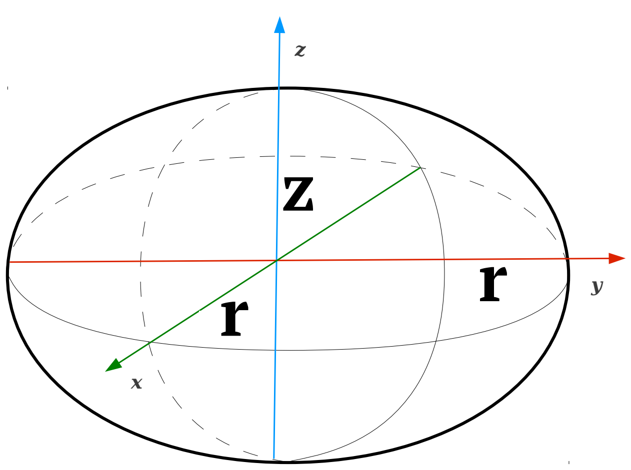
Prolate Spheroid

$\gamma = 1$



z=γr

$\gamma < 1$

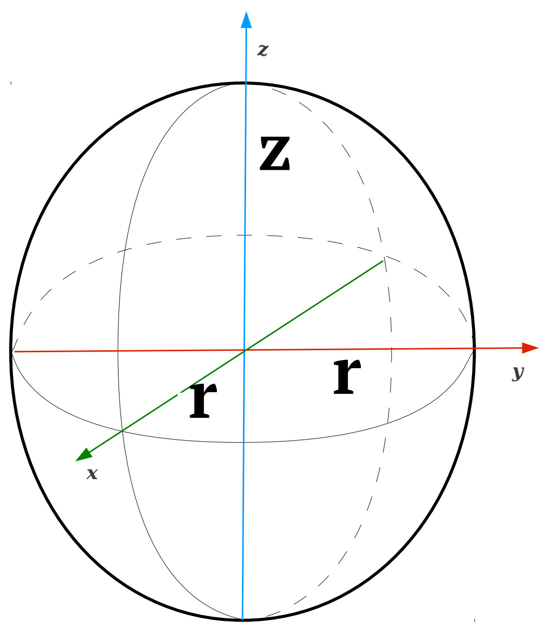


Oblate Spheroid

Parameterization

Using the parameterized metric, we make an anzats for a metric that will take the form[1,2]

$\gamma > 1$



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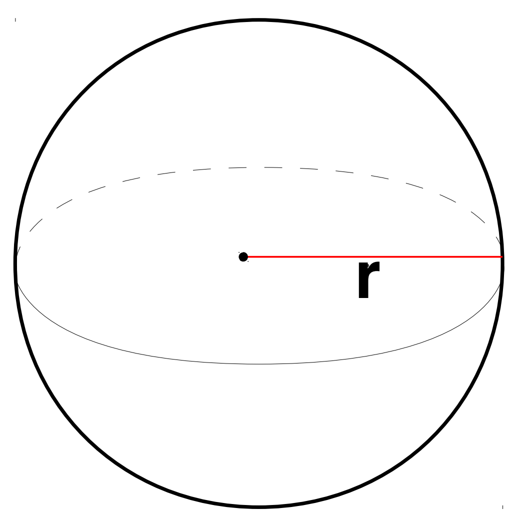
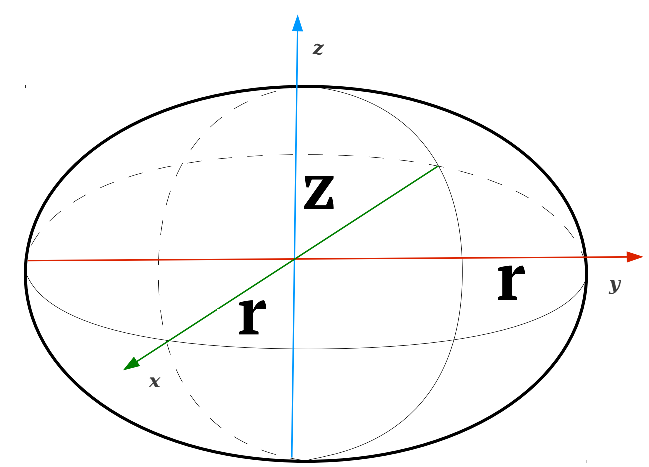


$z = \gamma r$

$\gamma < 1$



Oblate Spheroid

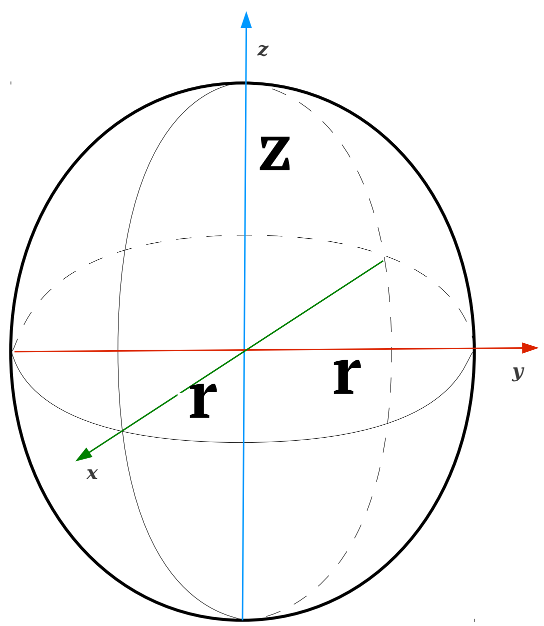
[1] O. Zubairi, A. Romero, and F. Weber, J. Phys. Conf.: Ser. **615**, 012003 (2015)
[2] O. Zubairi et al, Arxiv:1504-03006v1, (2015)

Parameterization

Using the parameterized metric, we make an anzats for a metric that will take the form[1,2]

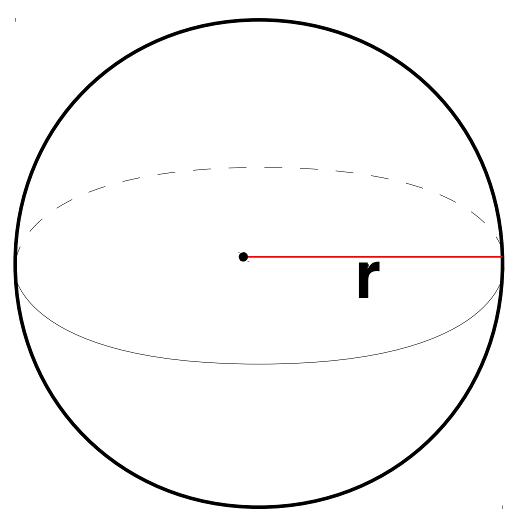
$$ds^2 = e^{2\Phi}dt^2 - \left(1 + \frac{2m(r)}{r}\right)^{-\gamma} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2$$

$\gamma > 1$



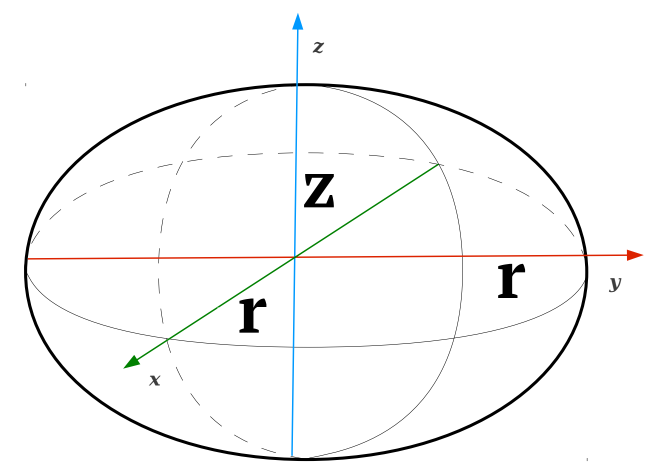
Prolate Spheroid

$\gamma = 1$



z=γr

$\gamma < 1$



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Einstein's Field Equations

$$T^{\mu}{}_{\nu} = (\epsilon + P)\, u^{\mu}\, u_{\nu} + g^{\mu}{}_{\nu}\, P \ ,$$

$$G^{t}_{t} \equiv R^{t}_{t} - \frac{1}{2} R = -T^{t}_{t} \ ,$$

$$G^{r}_{r} \equiv R^{r}_{r} - \frac{1}{2} R = -T^{r}_{r} \ ,$$

$$G^{\theta}_{\theta} \equiv R^{\theta}_{\theta} - \frac{1}{2} R = -T^{\theta}_{\theta} \ ,$$

$$G^{\phi}_{\phi} \equiv R^{\phi}_{\phi} - \frac{1}{2} R = -T^{\phi}_{\phi}$$

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2} g^{\mu}{}_{\nu} = -8\pi T^{\mu}{}_{\nu}$$

Use together

$$ds^2 = \mathrm{e}^{2\Phi} dt^2 - \left(1 + \frac{2m(r)}{r}\right)^{-\gamma} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2$$

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$$G^t_t \equiv R^t_t - \frac{1}{2}R = -T^t_t \ ,$$

$$G^r_r \equiv R^r_r - \frac{1}{2}R = -T^r_r \ ,$$

$$G^\theta_\theta \equiv R^\theta_\theta - \frac{1}{2}R = -T^\theta_\theta \ ,$$

$$G^\phi_\phi \equiv R^\phi_\phi - \frac{1}{2}R = -T^\phi_\phi$$

Use together

$$ds^2 = \mathrm{e}^{2\Phi} dt^2 - \left(1 + \frac{2m(r)}{r}\right)^{-\gamma} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2$$

Will have to recalculate Ricci tensor & the Ricci scalar...

Einstein's Field Equations

The non-vanishing Christoffel symbols are calculated to be:

$$\Gamma^{a}{}_{\mu\nu} = \frac{1}{2} g^{ab} \left(\frac{\partial g_{b\mu}}{\partial x^{\nu}} + \frac{\partial g_{b\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{b}} \right)$$

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$$\Gamma^{r}{}_{tt} = \beta\ \mathrm{e}^{2\Phi(r)} \Phi'(r)\ , \Gamma^{t}{}_{tr} = \Phi'(r)\ ,$$

$$\Gamma^{r}{}_{rr} = \frac{\gamma\,[-m'(r)r + m(r)]}{r[-r + 2m(r)]}\ , \Gamma^{\theta}{}_{r\theta} = \Gamma^{\Phi}{}_{r\Phi} = \frac{1}{r}\ ,$$

$$\Gamma^{r}{}_{\theta\theta} = -\beta\ r\ , \Gamma^{\Phi}{}_{\Phi\Phi} = \cot(\theta)\ ,$$

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where the primes denote derivatives with respect to the radial coordinate, ***r***, and

$$\beta \equiv \left(\frac{r - 2m(r)}{r} \right)^{\gamma}$$

Einstein's Field Equations

$$R_t^t = \frac{1}{r(r-2m(r))}\Big[\beta\Phi'(r)m'(r)\gamma r - \Phi'(r)m(r)\gamma - (\Phi'(r))^2 rm(r)$$
$$- \Phi''(r)r^2 + 2\Phi''(r)rm(r) - 2\Phi'(r)r + 4\Phi'(r)m(r)\Big]$$

$$R_r^r = \frac{1}{r^2(r-2m(r))}\Big[-\beta\Phi''(r)r^3 - 2\Phi''(r)r^2m(r) + (\Phi'(r))^2 r^3 - 2(\Phi'(r))^2 r^2 m(r)$$
$$- \gamma\Phi'(r)m'(r)r^2 + \gamma\Phi'(r)rm(r) - 2\gamma m'(r)r + 2\gamma m(r)\Big]$$

Ricci Tensor Components

$$R_\theta^\theta = \frac{1}{r^2(r-2m(r))}\Big[-\beta r^2\Phi'(r) + 2\beta\gamma r\Phi'(r)m(r) + \beta\gamma m'(r)r - \beta\gamma m(r)$$
$$+ r - 2m(r) - \beta\gamma + 2\beta m(r)\Big]$$

$$R_\phi^\phi = R_\theta^\theta$$

Ricci Scalar:

$$R = \frac{2}{r^2(r-2m(r))}\Big[\beta\gamma\Phi'(r)m'(r)r^2 - \beta\gamma\Phi'(r)m(r)r - \beta(\Phi'(r))^2 r^3$$
$$+ 2\beta(\Phi'(r))^2 r^2 m(r) - \beta\Phi''(r)r^3 + 2\beta\Phi''(r)r^2m(r) - 2m(r)$$
$$- 2\beta\gamma^2\Phi'(r) + 4\beta\Phi'(r)m(r) + 2\beta\gamma m'(r)r - 2\beta\gamma m(r) + r$$
$$- 2m(r) - \beta\gamma + 2\beta m(r)\Big]$$

$$\beta \equiv \left(\frac{r-2m(r)}{r}\right)^\gamma$$

Einstein's Field Equations

Substitute Ricci tensor components & Ricci Scalar into the field equations:

And Solve!!

$$G^t_t \equiv R^t_t - \frac{1}{2}R = -T^t_t \ ,$$

$$G^r_r \equiv R^r_r - \frac{1}{2}R = -T^r_r \ ,$$

$$G^\theta_\theta \equiv R^\theta_\theta - \frac{1}{2}R = -T^\theta_\theta \ ,$$

$$G^\phi_\phi \equiv R^\phi_\phi - \frac{1}{2}R = -T^\phi_\phi$$

Modified Hydrostatics

The stellar structure equation is derived to be[1,2]:

$$\frac{dP}{dr} = -(\epsilon + P)\frac{\left[\frac{1}{2}r + 4\pi r^3 P - \frac{1}{2}r\left(1 - \frac{2m}{r}\right)^{\gamma}\right]}{r^2\left(1 - \frac{2m}{r}\right)^{\gamma}}$$

$\epsilon = \text{Energy Density}, \quad P = \text{Pressure}, \quad m = \text{mass}, \quad r = \text{radial distance}$

$\gamma = \text{Deformation Constant}$

Total Mass of our mass distribution is: $M = m(r)$

where *r* is defined to be when $P(r = R) = 0$

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Still need take Polar direction into account & parameterize.

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Herrera L. et al 1999 → $M = \gamma m$

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Modified Hydrostatics

Volume of sphere:

$$V = \frac{4}{3}\pi r^3 \implies V = \frac{4}{3}\pi r^2 z \text{ (Spheroid)}$$

Parameterize *z-component*:

$$z = \gamma r \longrightarrow V = \frac{4}{3}\pi r^3 \gamma$$

Total Mass

$$\frac{dm}{dr} = 4\epsilon\pi r^2 \gamma$$

Stellar Structure

$$\frac{dP}{dr} = -(\epsilon + P)\frac{\left[\frac{1}{2}r + 4\pi r^3 P - \frac{1}{2}r\left(1 - \frac{2m}{r}\right)^{\gamma}\right]}{r^2\left(1 - \frac{2m}{r}\right)^{\gamma}}$$

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- However, we must also impose (*Paret, Horvath, & Martinez, 2014*)

$$P(z = Z) = 0$$

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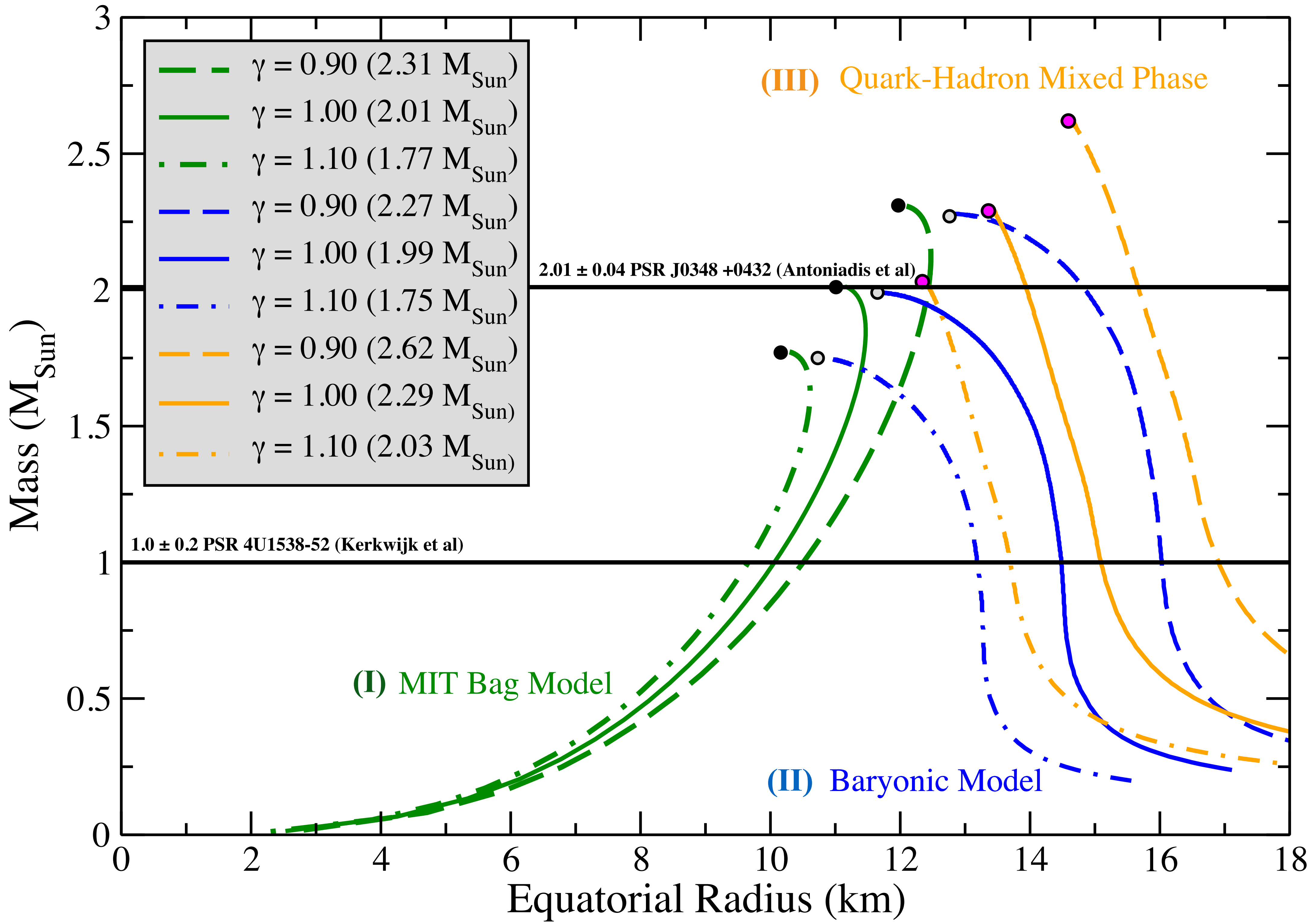
$$P(z = Z) = 0$$

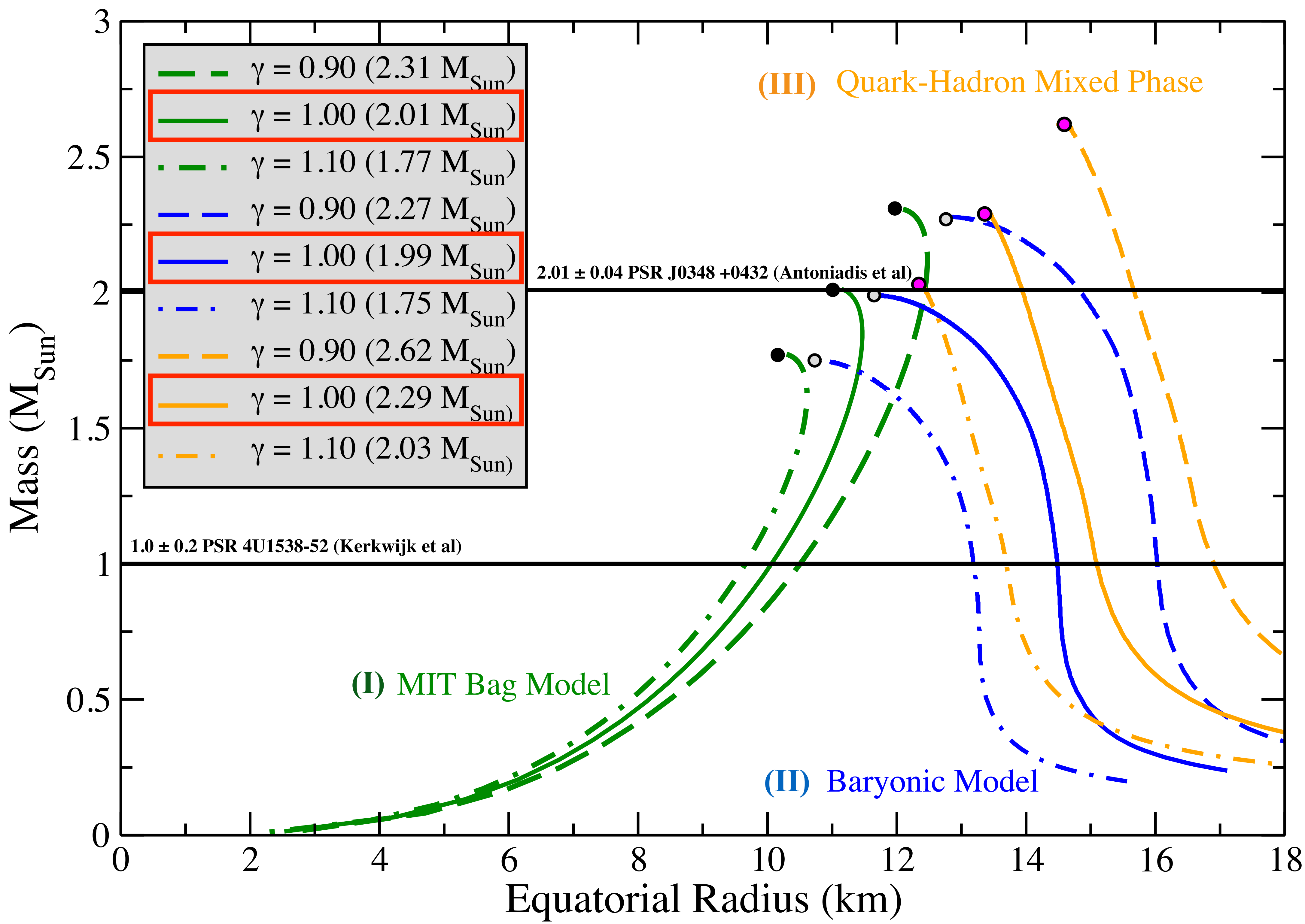
$$P(r = R) = 0$$

Mass-Radius Relations



Zubairi et al, 2017

Mass-Radius Relations



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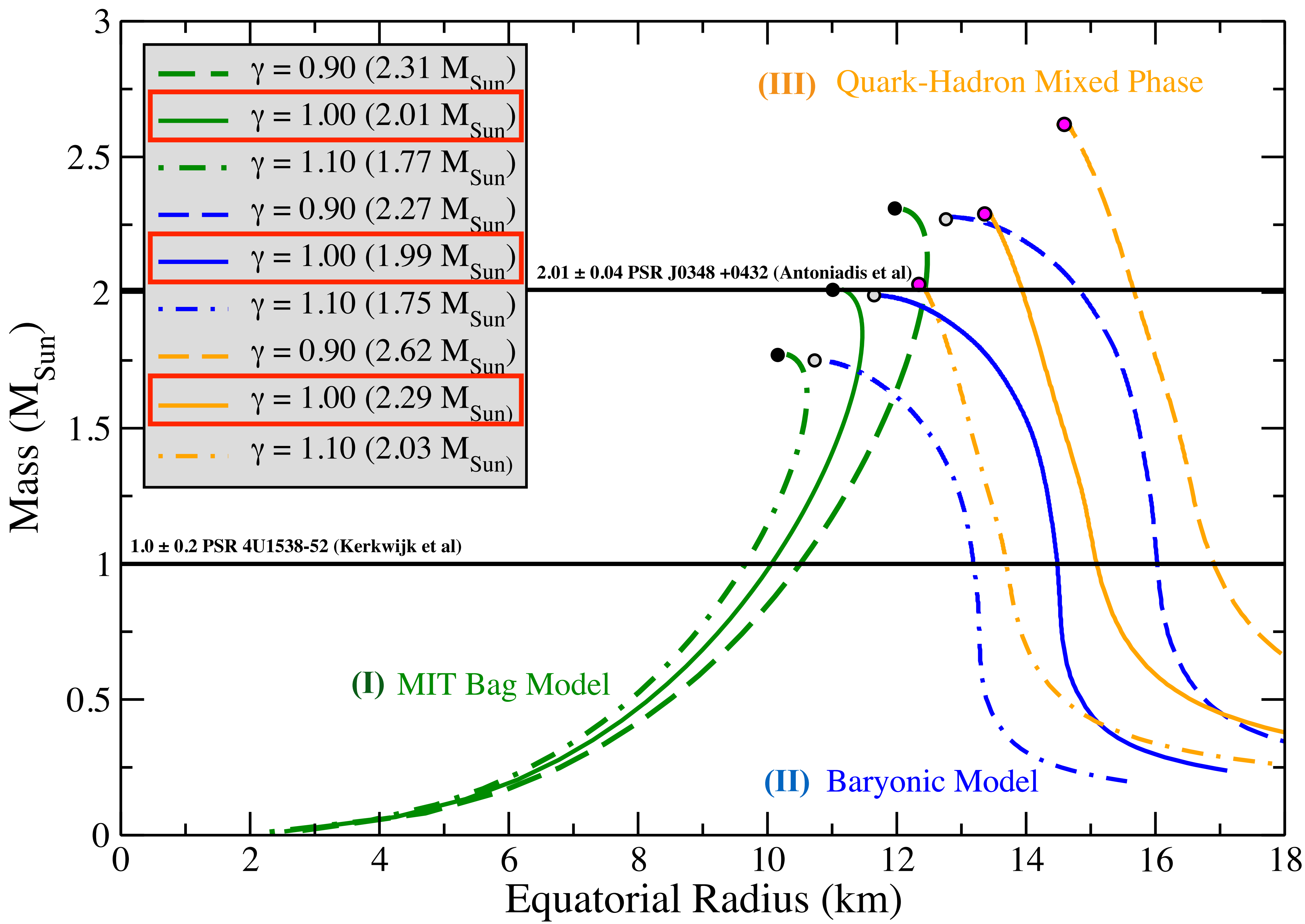

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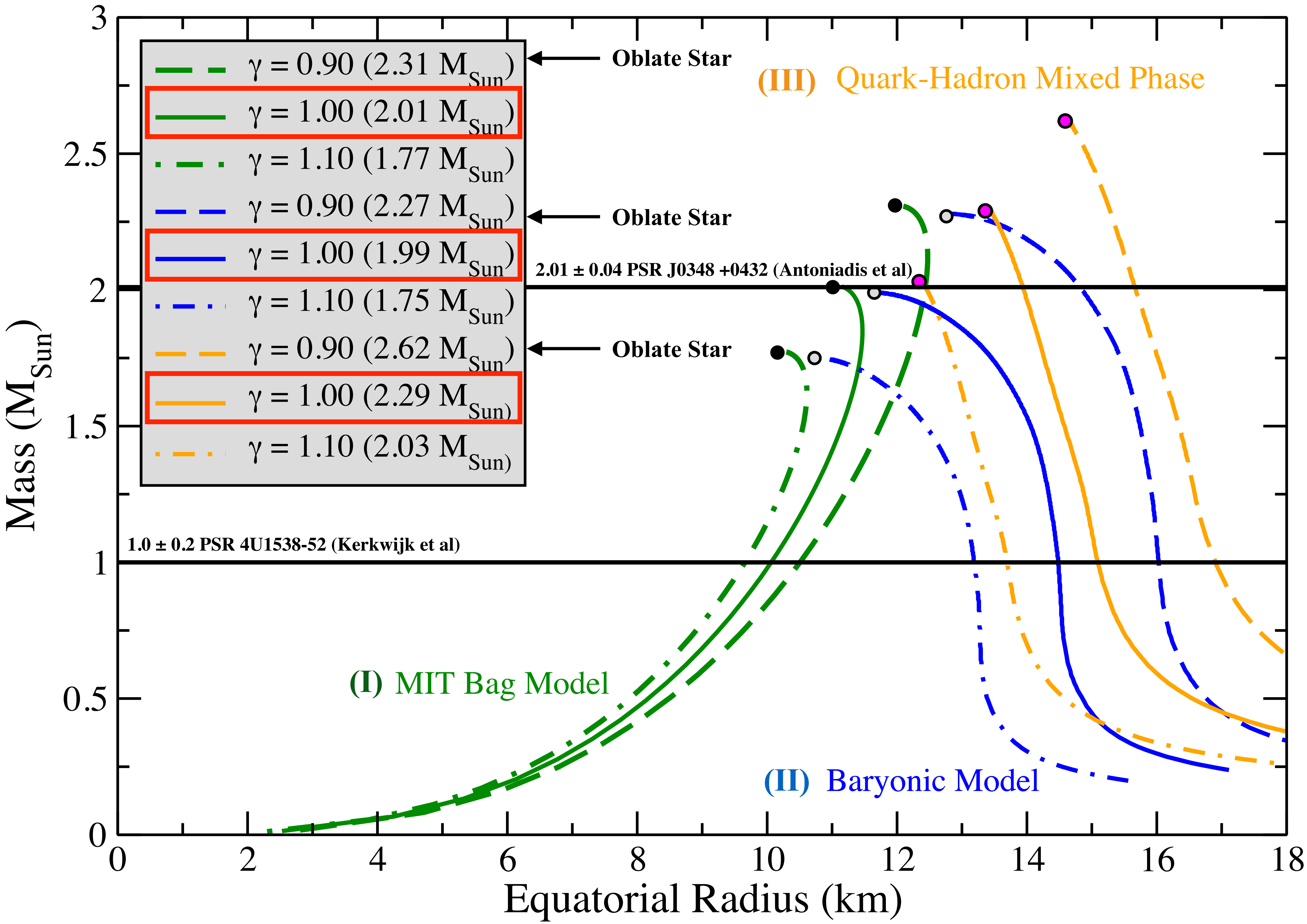

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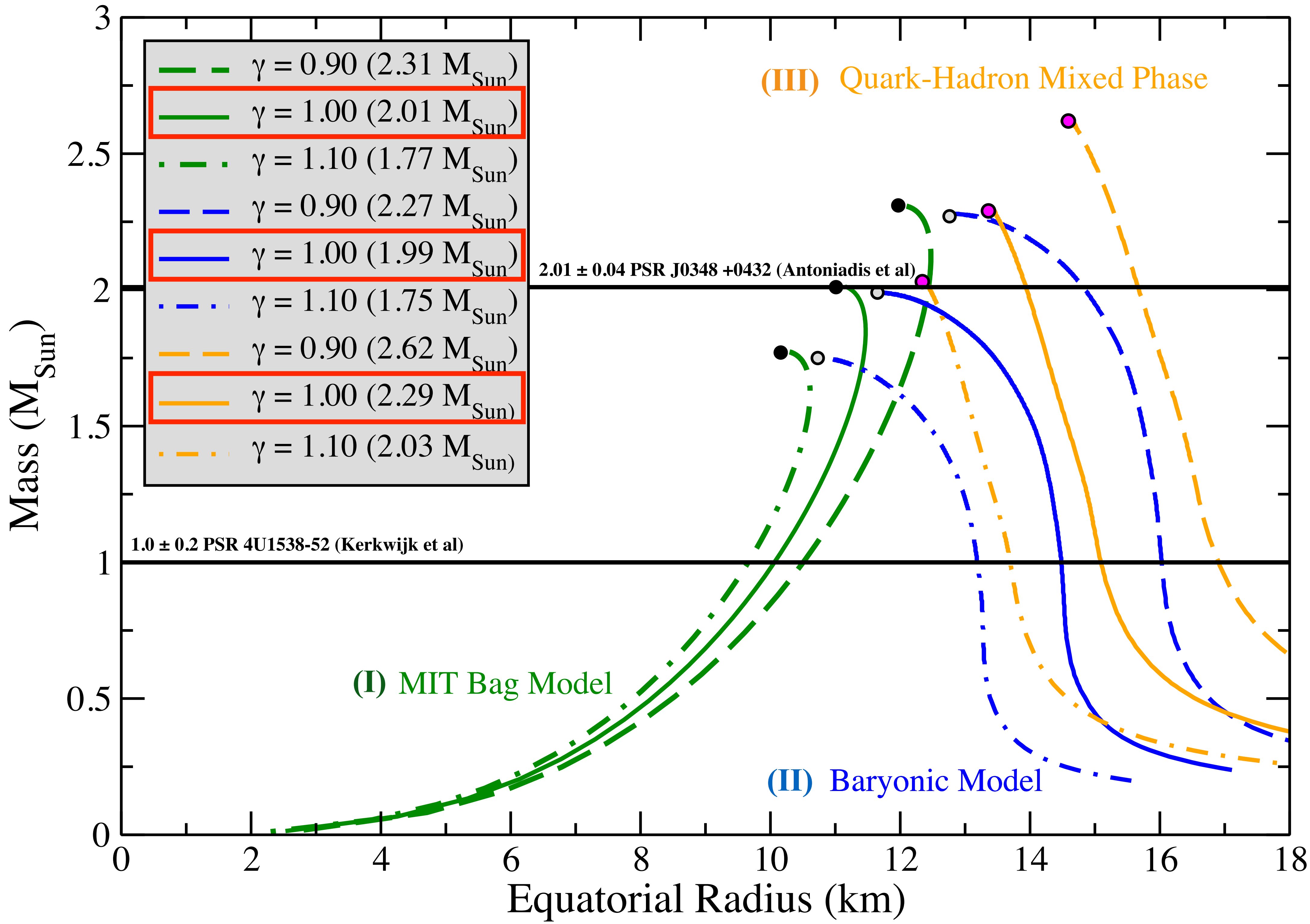
Mass-Radius Relations



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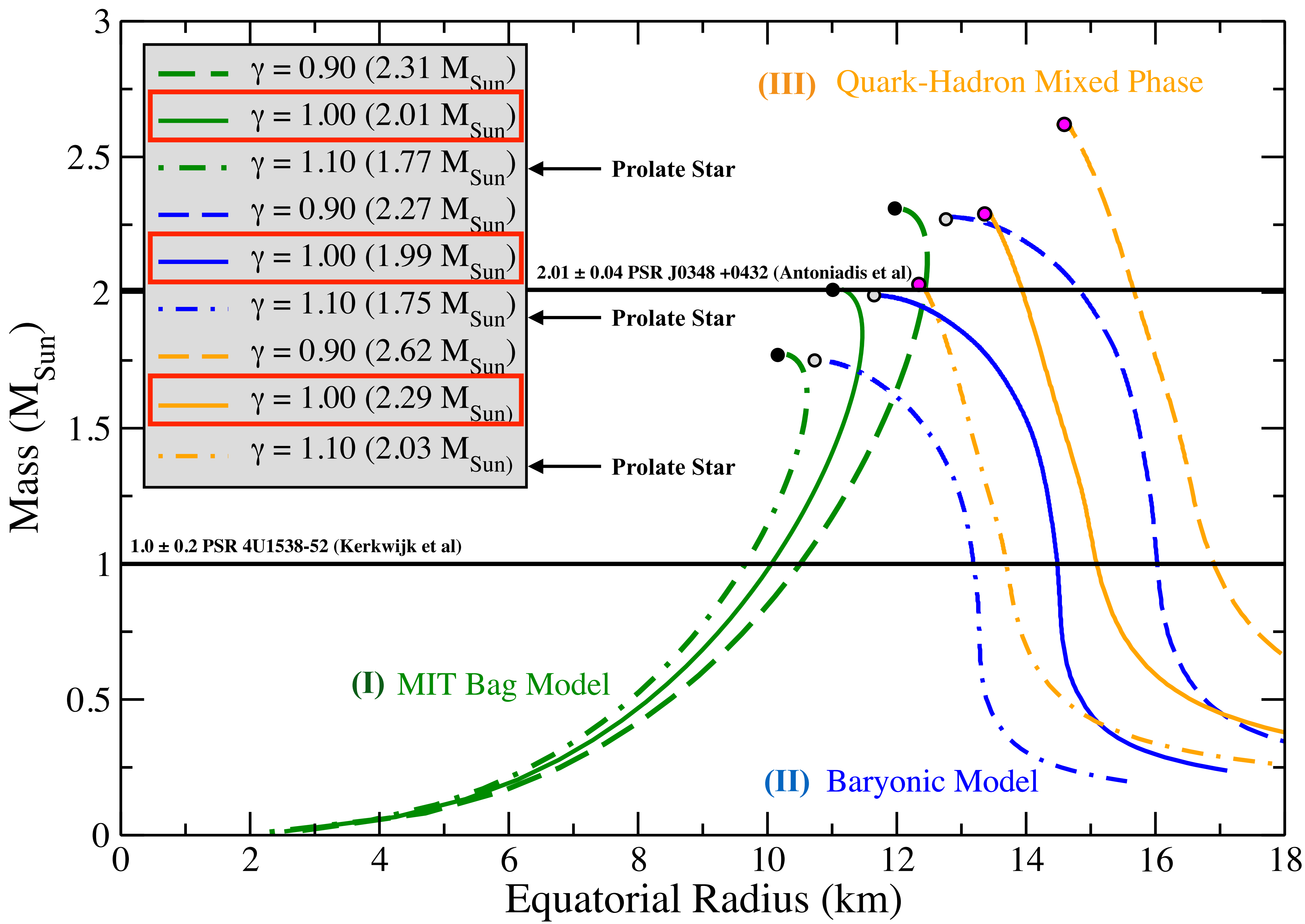
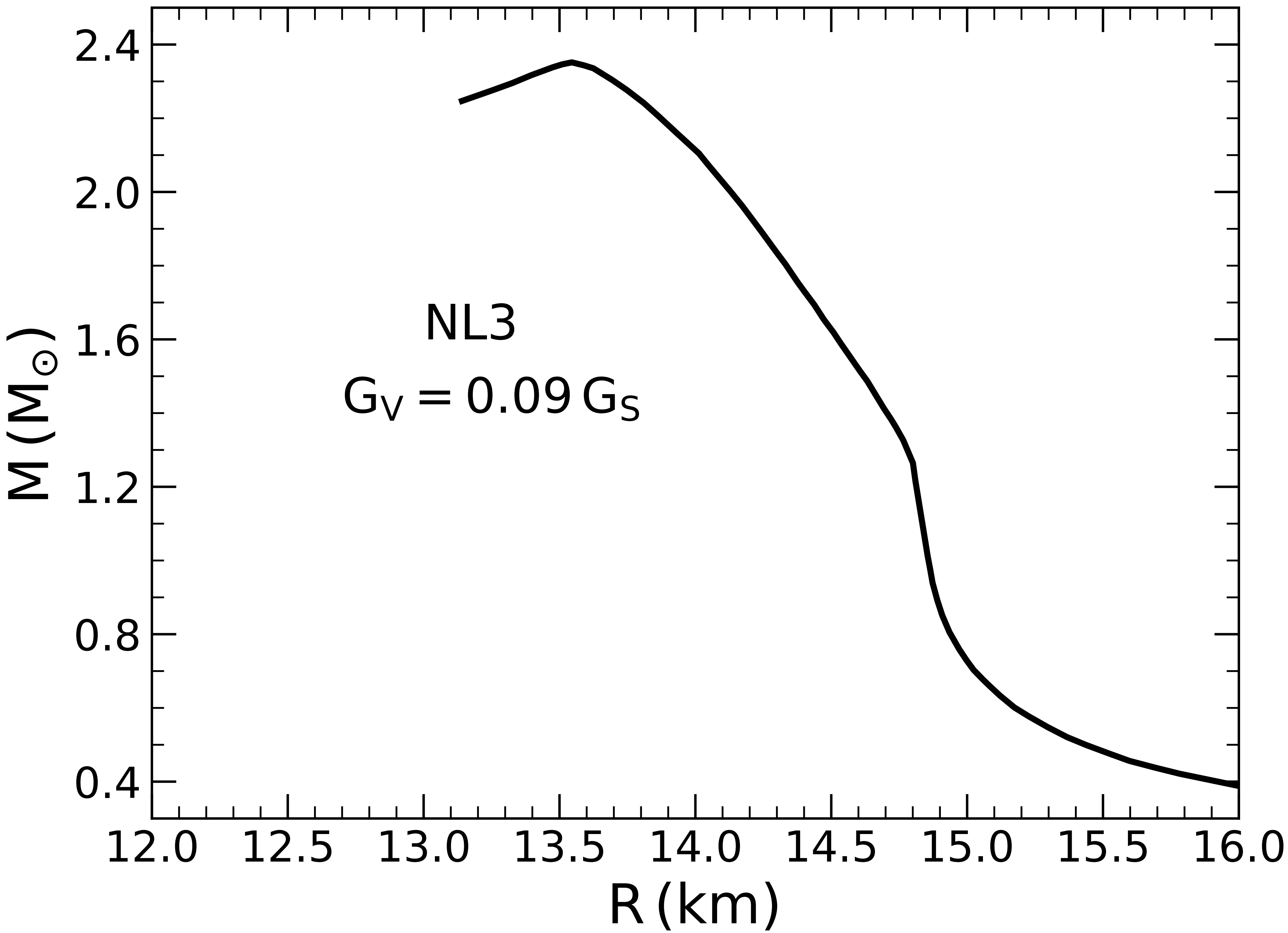
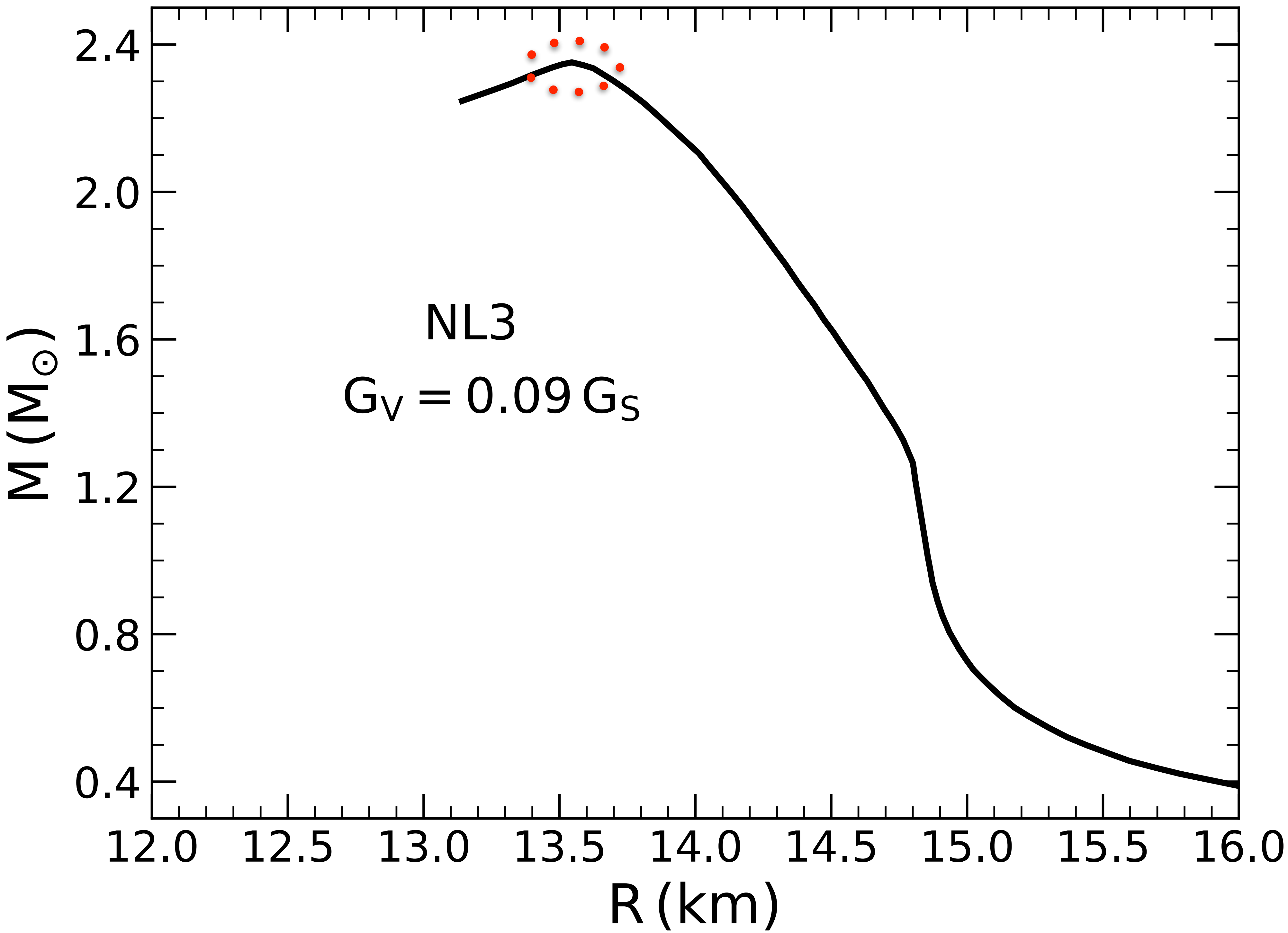


Zubairi et al, 2017

Mass-Radius Relations Model III



Orsaria M. et al., Phys. Rev. C, ***89*** *(2014)*



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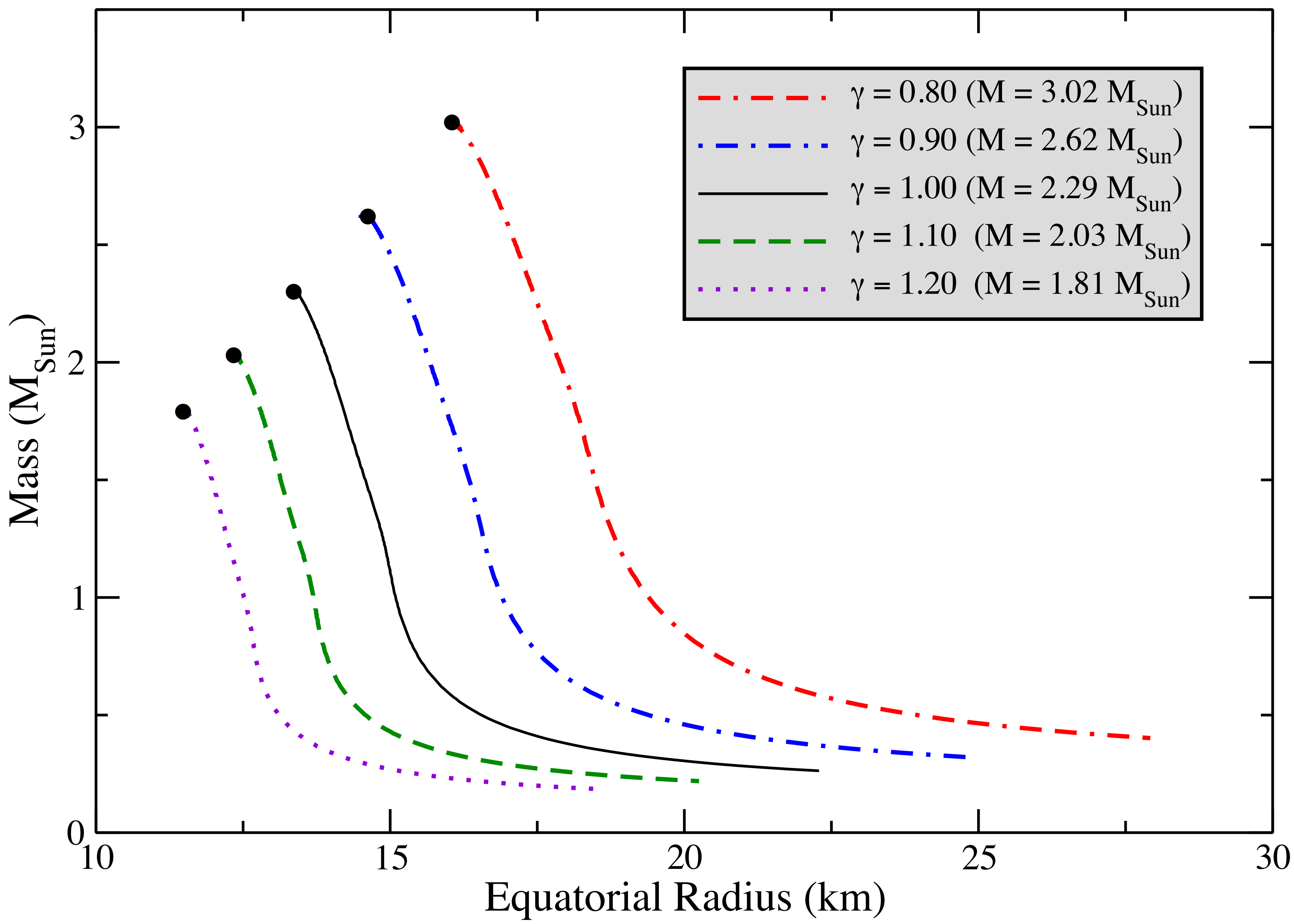
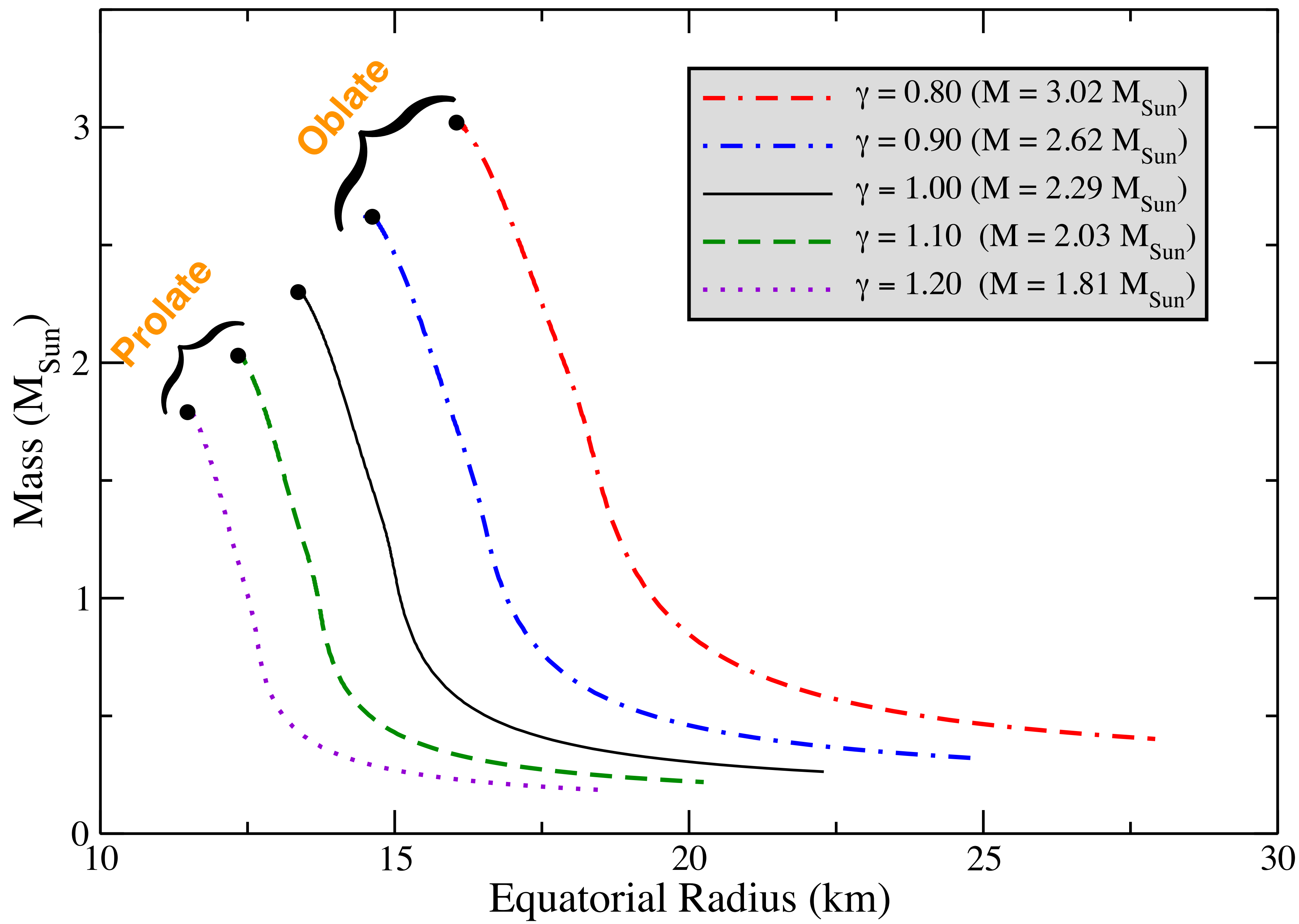
Mass-Radius Relations

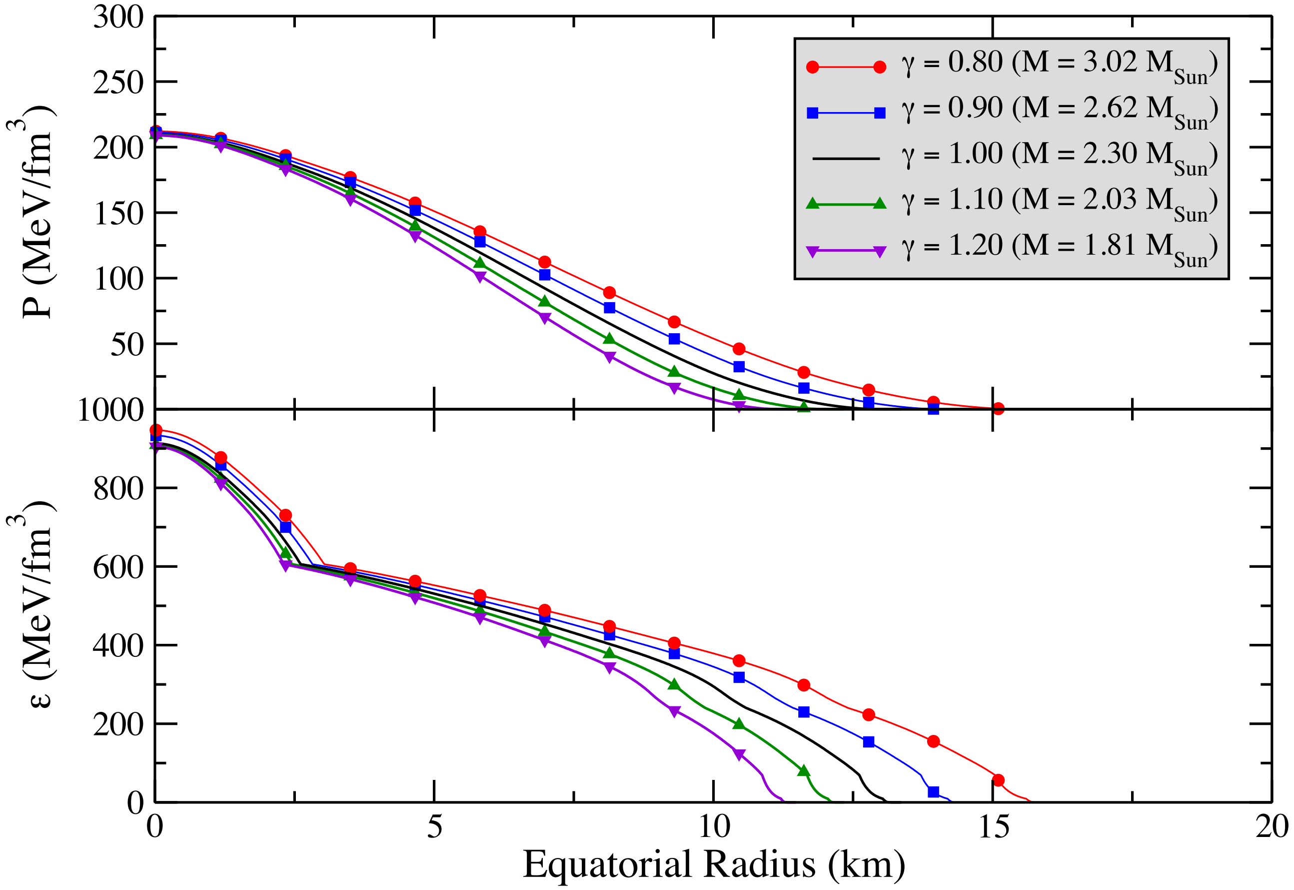


Zubairi et al, 2015

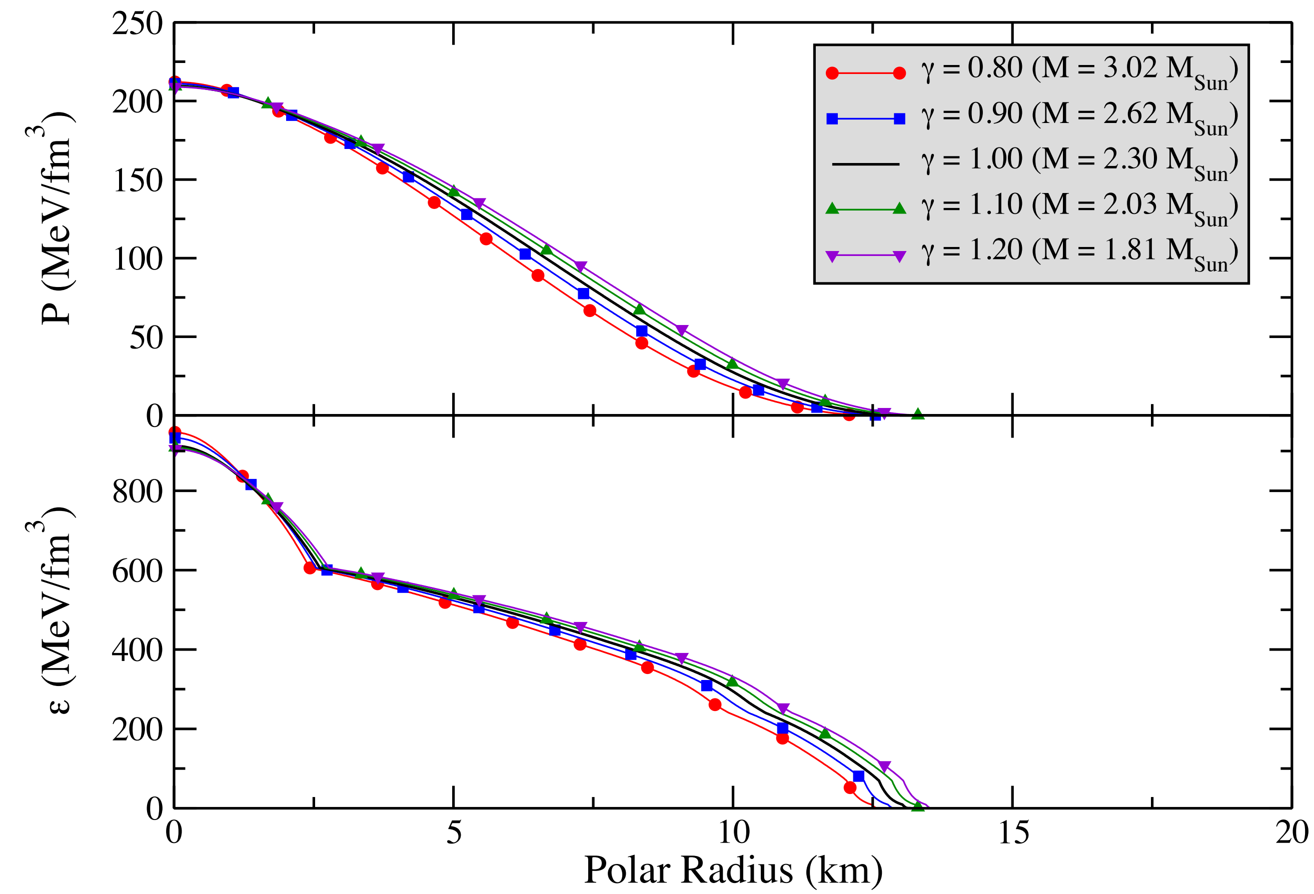


Zubairi et al, 2015

Pressure & Energy - Density Profiles Model III



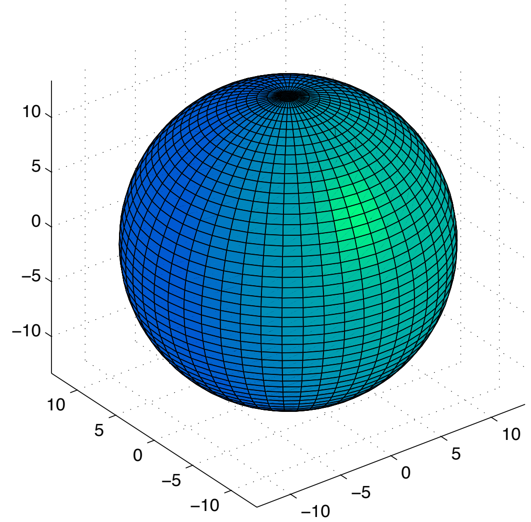
Polar Radius



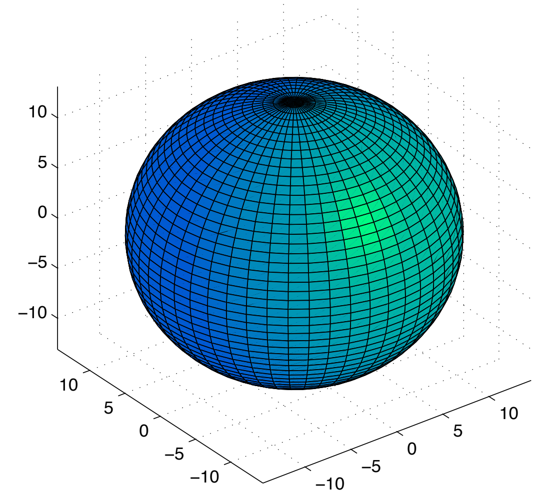
Equatorial Radius

Zubairi et al, 2015

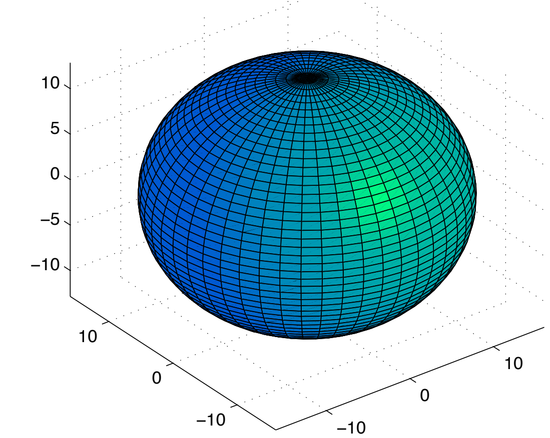
γ=1.00
M=2.30 $M_\odot$



γ=0.90
M=2.62 $M_\odot$

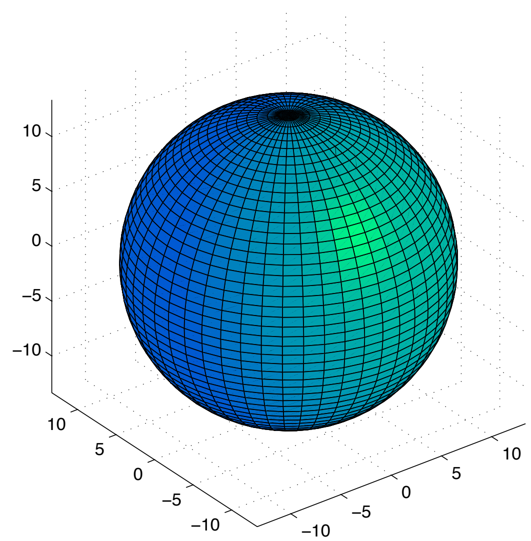


γ=0.80
M=3.02 $M_\odot$

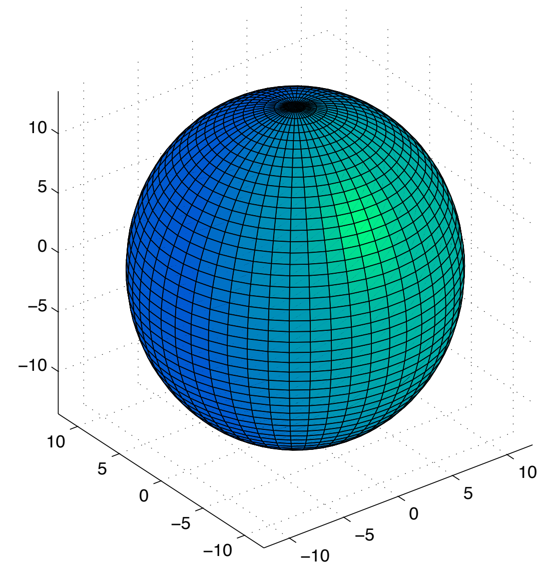


Oblate Case

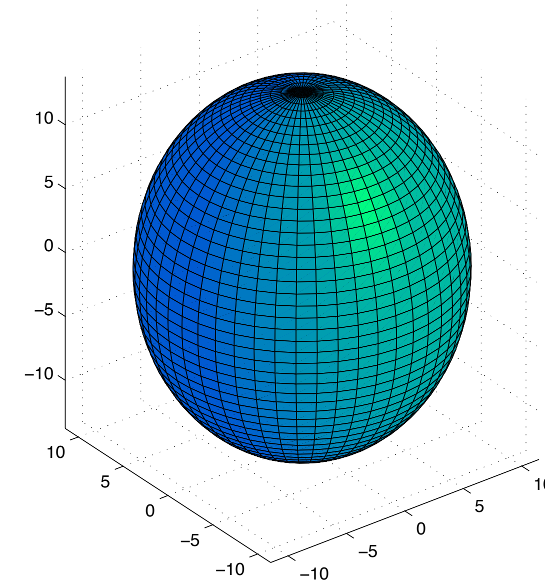
γ=1.00
M=2.30 $M_\odot$



γ=1.10
M=2.03 $M_\odot$

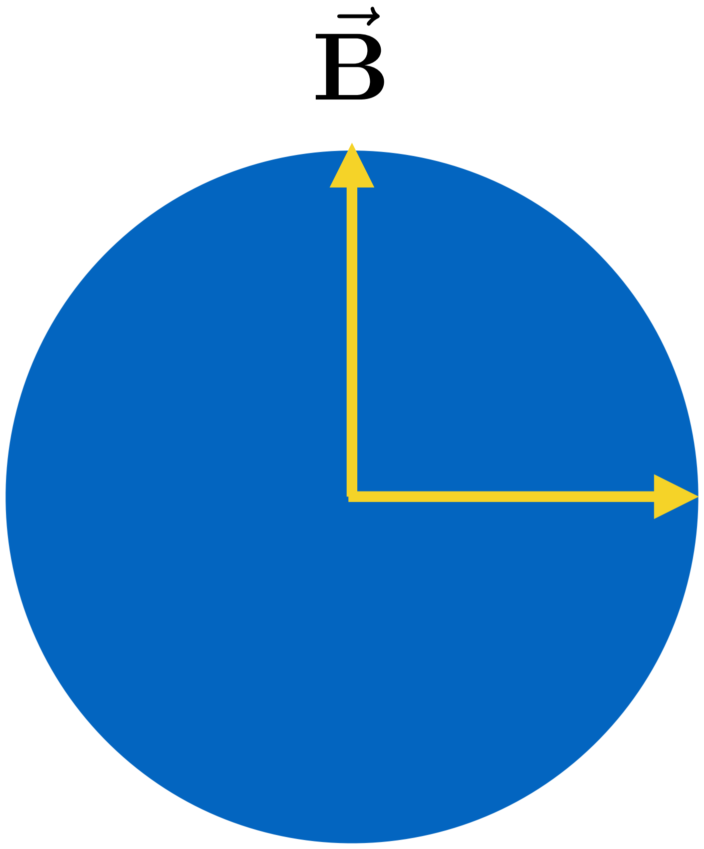


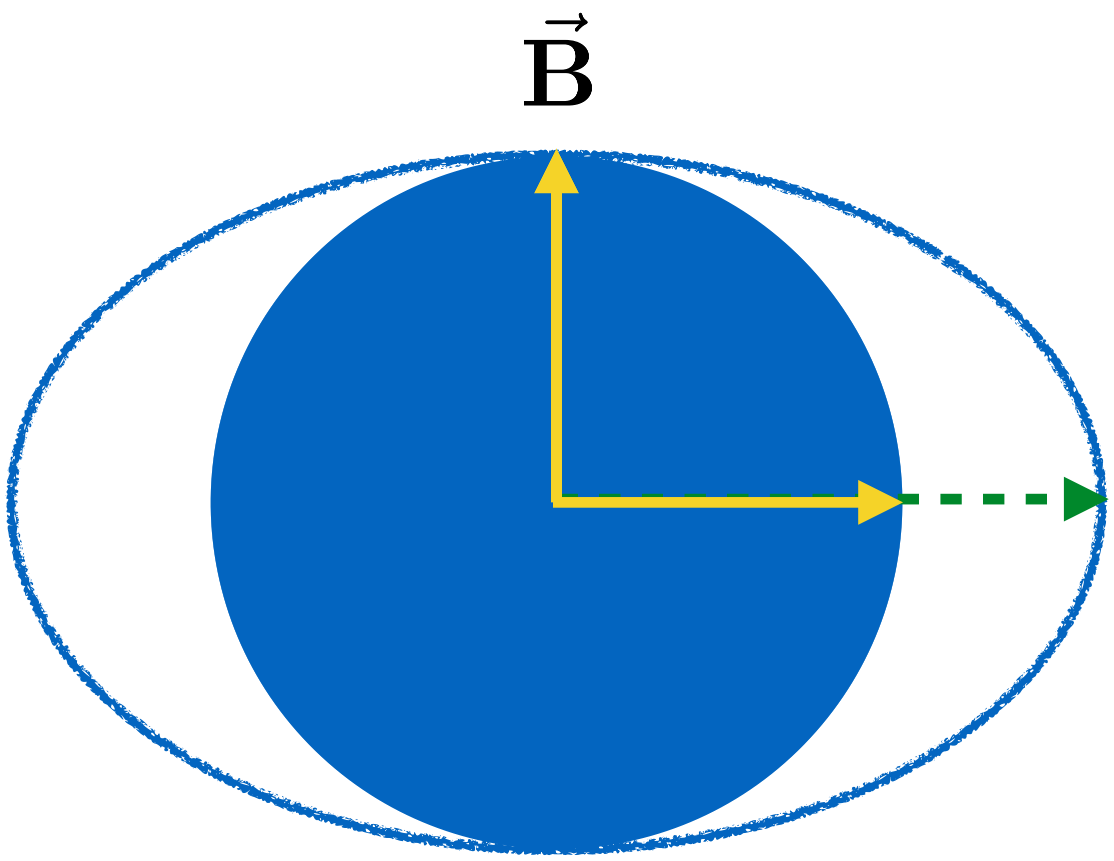
γ=1.20
M=1.81 $M_\odot$



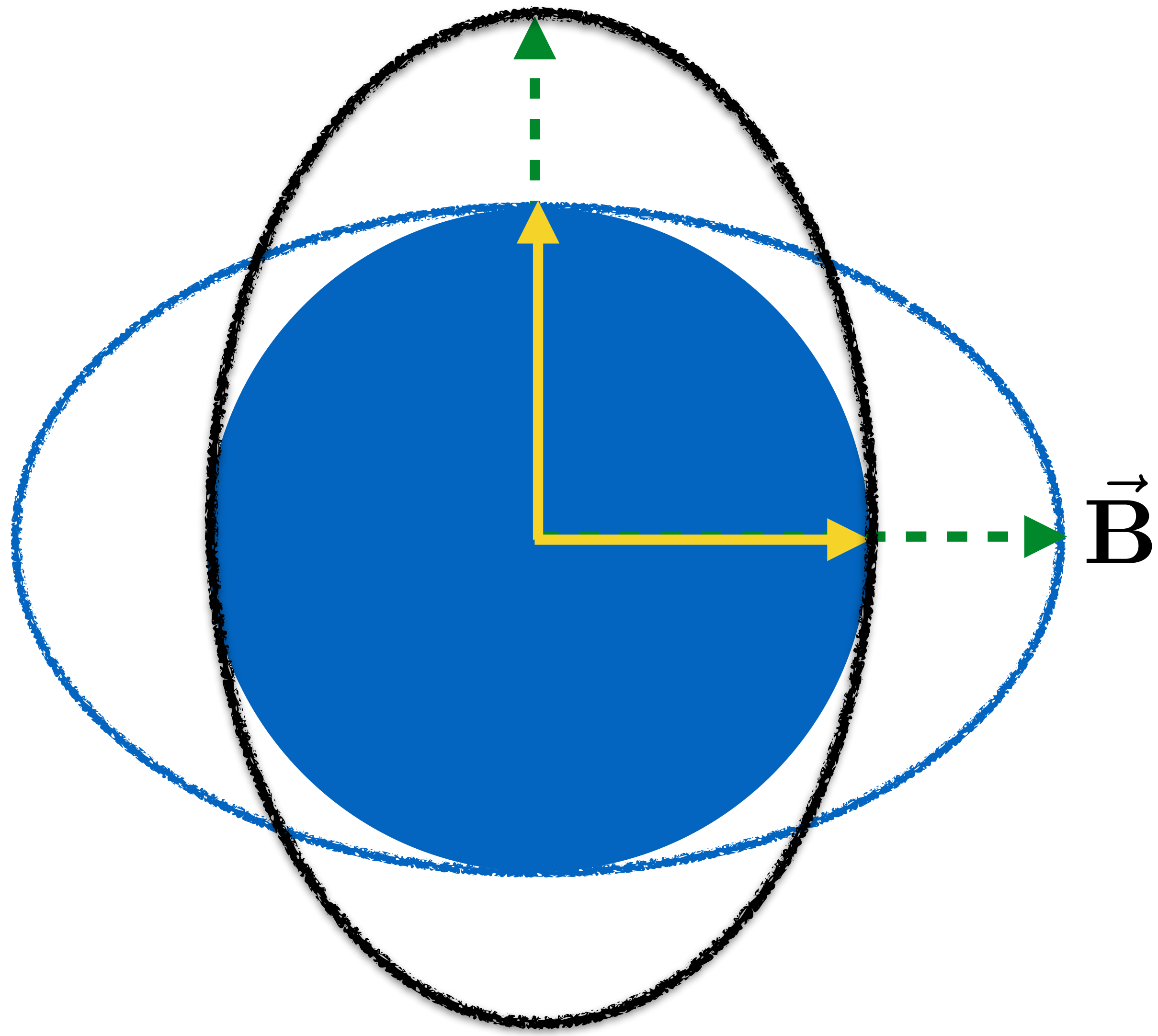
Prolate Case

Deformity




$\vec{B}$

Oblate Spheroids



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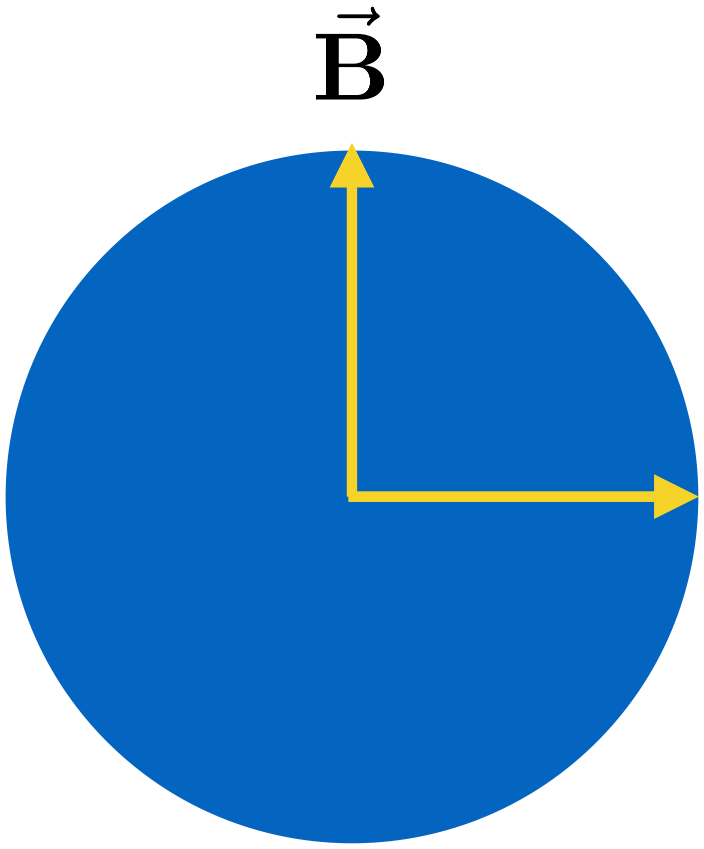

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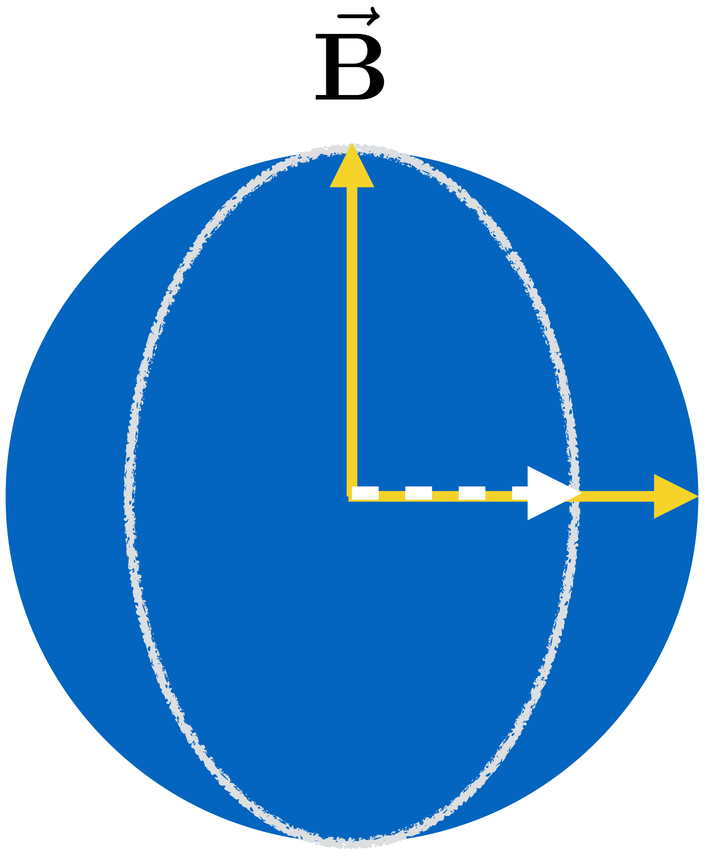
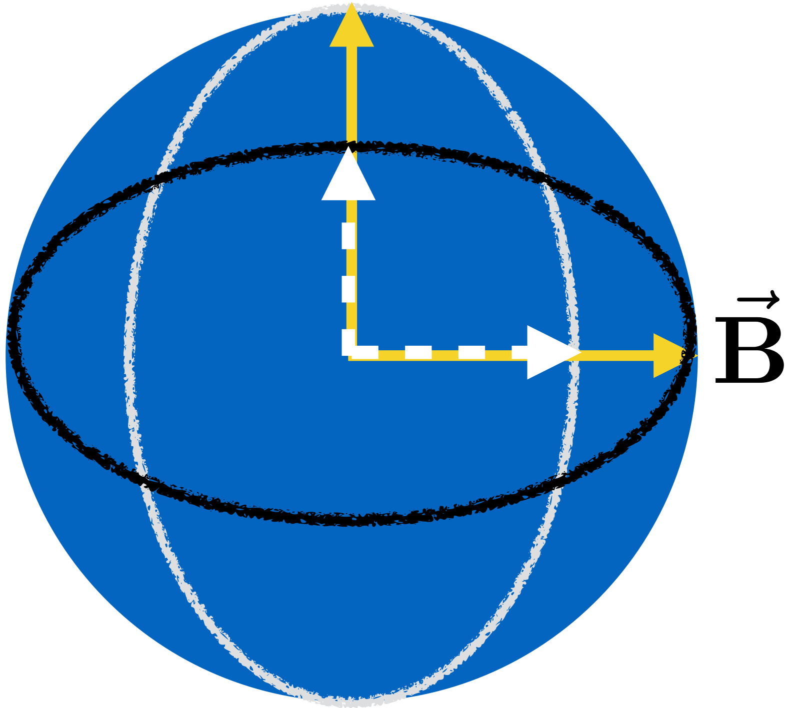
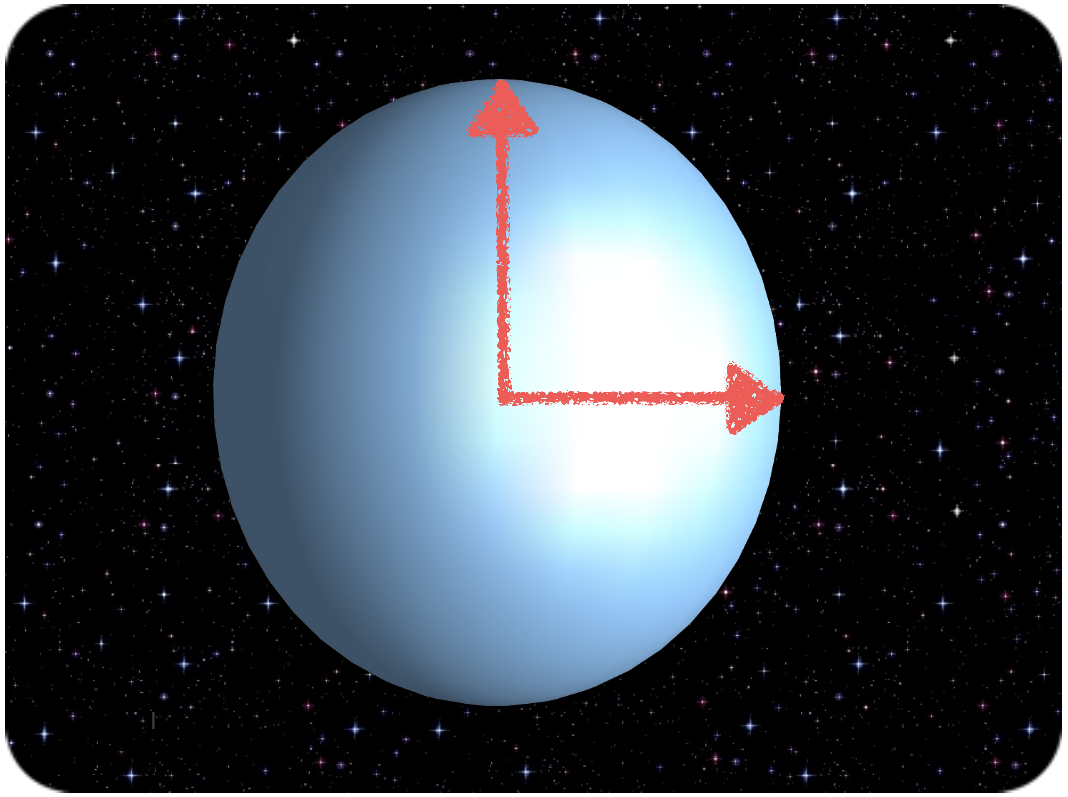


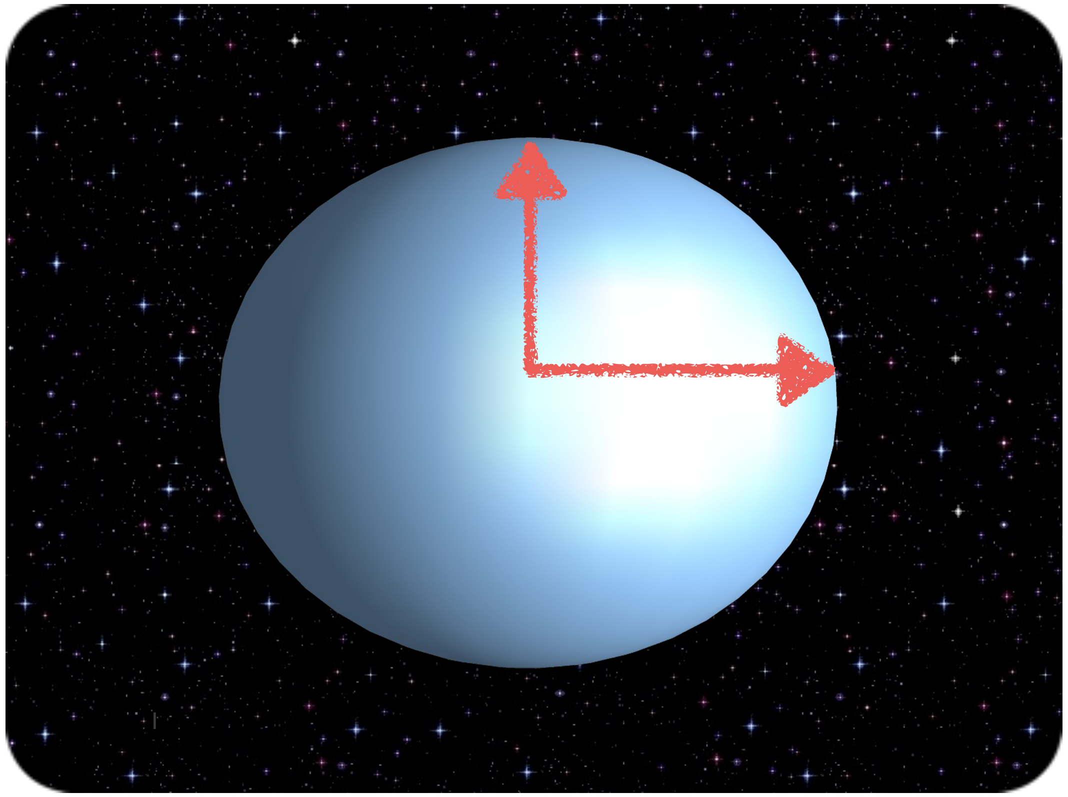

$\vec{B}$

Gravitational Mass-Quadrupole

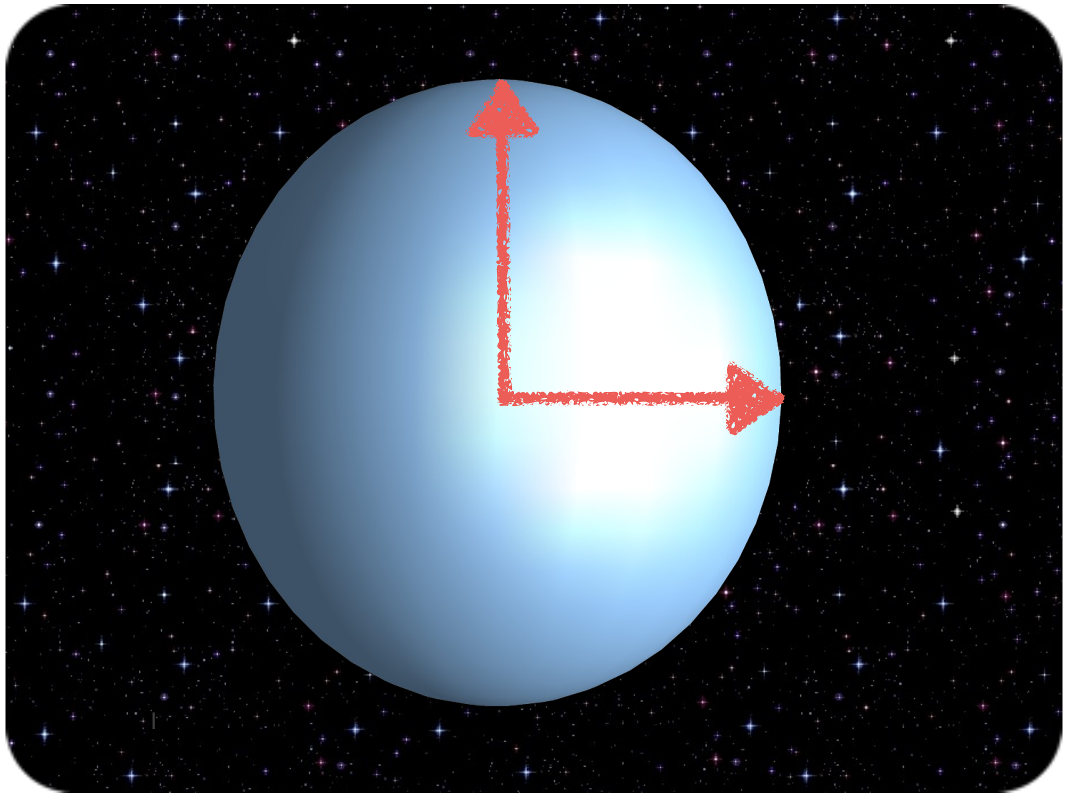


Mass

Homogeneity?



Gravitational Mass-Quadrupole



Mass

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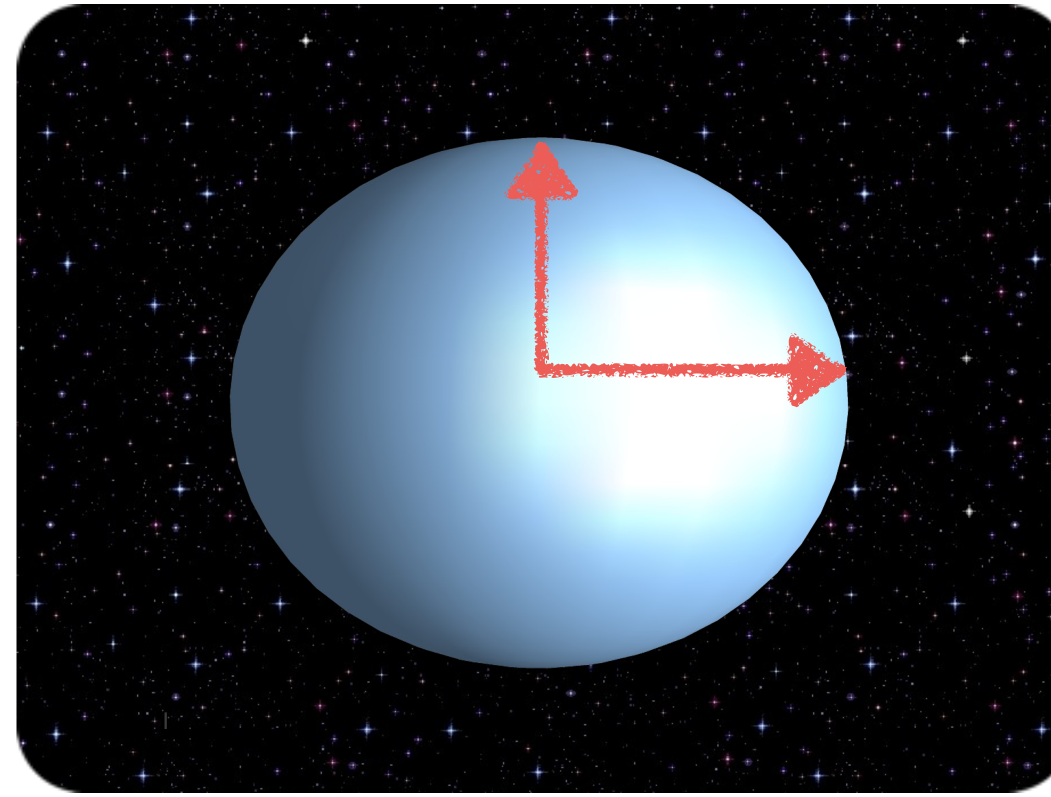
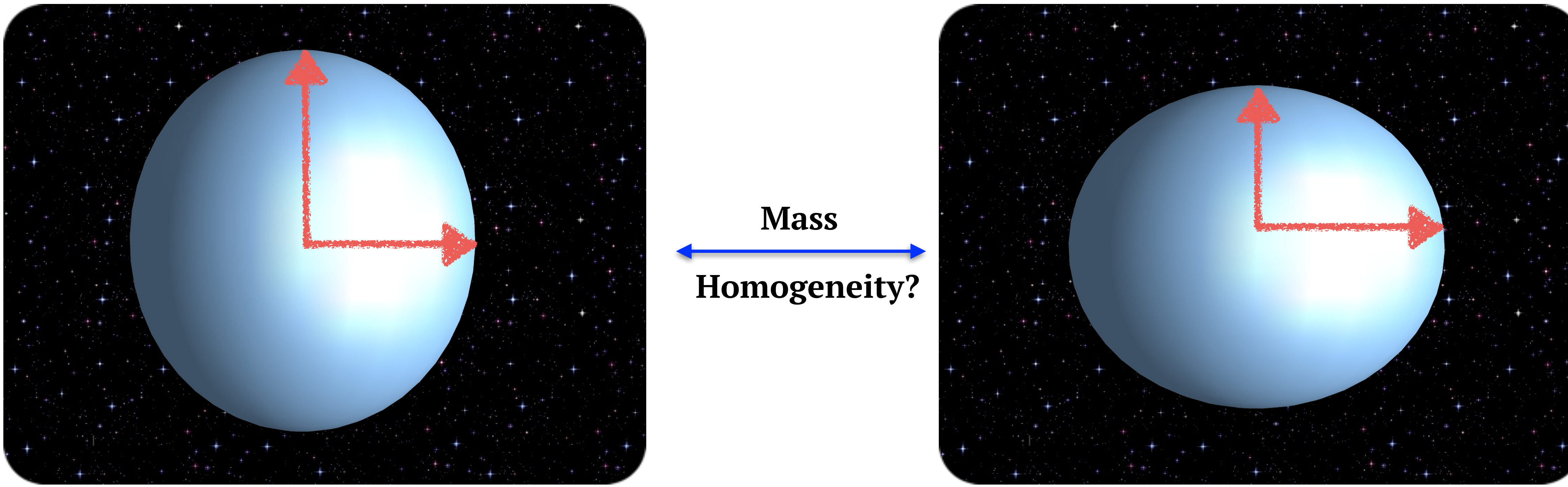


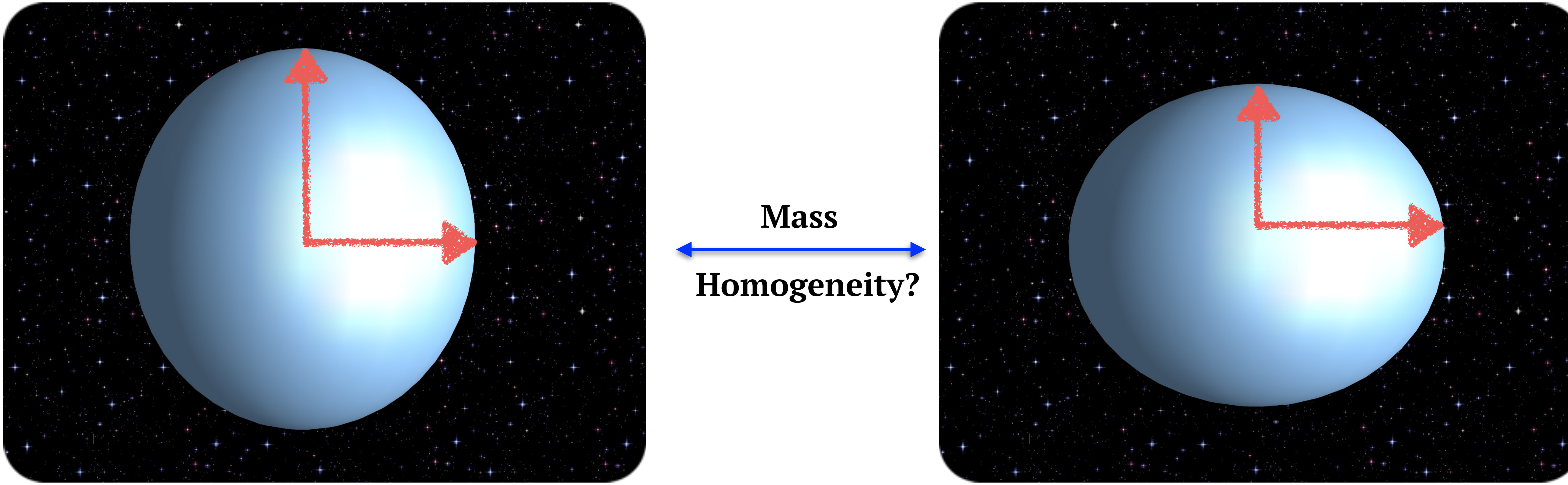
- The mass quadrupole moment (QM) is expected to be non-zero…

Gravitational Mass-Quadrupole



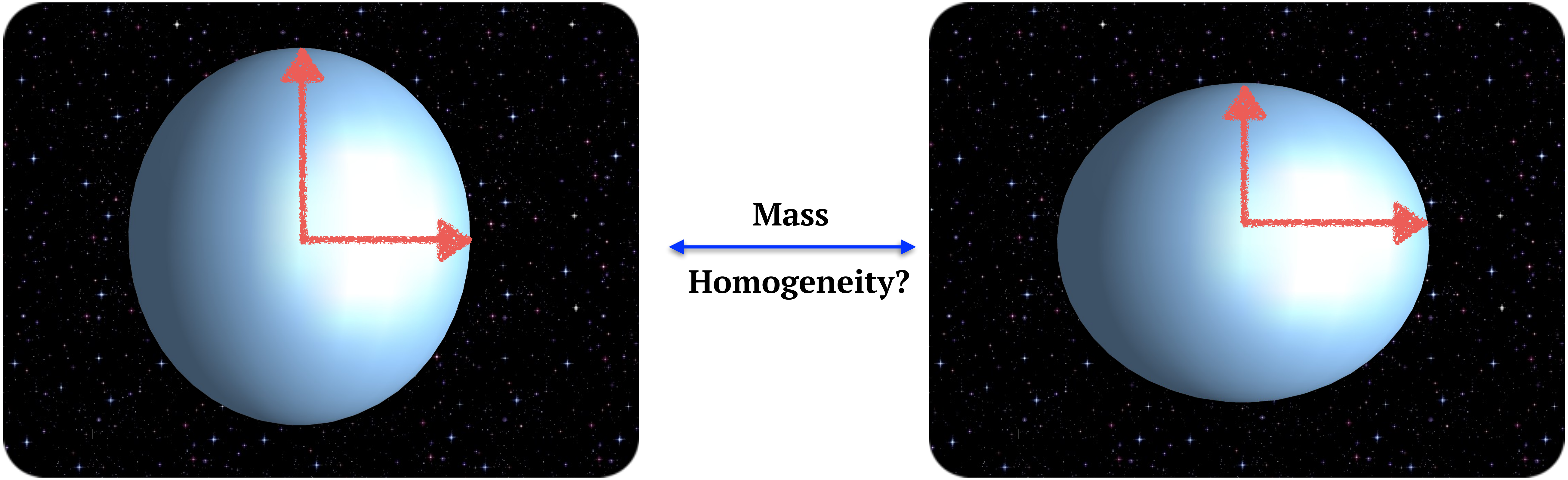
- The mass quadrupole moment (QM) is expected to be non-zero…
- Indicating the mass is not evenly distributed throughout the deformed object…

Gravitational Mass-Quadrupole



- The mass quadrupole moment (QM) is expected to be non-zero…
- Indicating the mass is not evenly distributed throughout the deformed object…
- The QM should **NOT** be symmetric and **NOT** be the same for prolate and oblate stars..

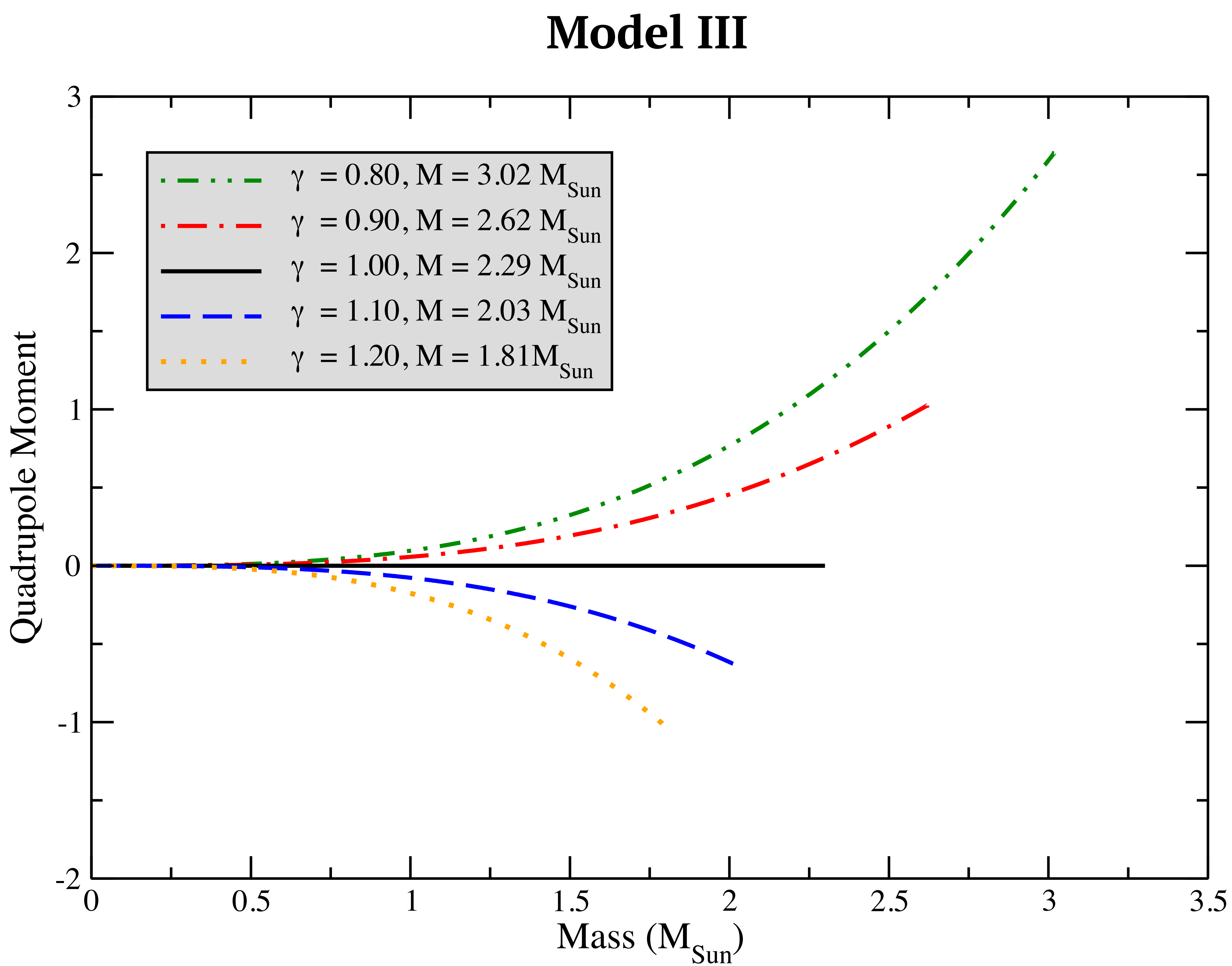
Gravitational Mass-Quadrupole



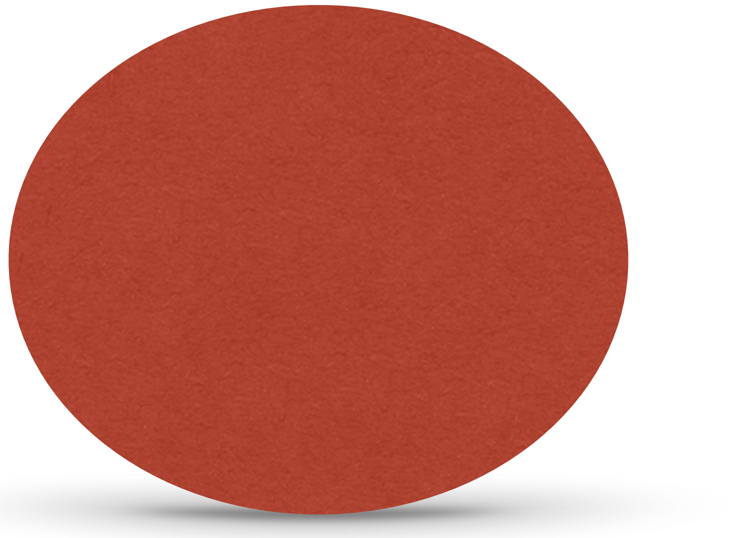
- The mass quadrupole moment (QM) is expected to be non-zero…
- Indicating the mass is not evenly distributed throughout the deformed object…
- The QM should **NOT** be symmetric and **NOT** be the same for prolate and oblate stars..

$$Q = \frac{\gamma}{3} M^3 \left(1 - \gamma^2\right)$$

Gravitational Mass-Quadrupole

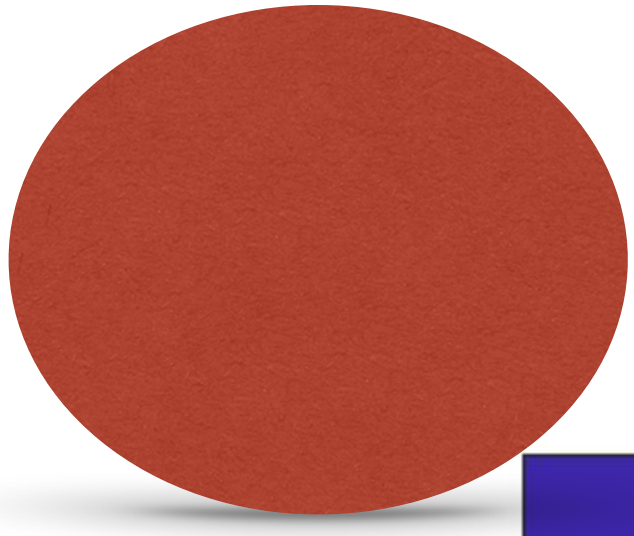


Mass Inhomogeneity Oblate Stars (γ=0.8)

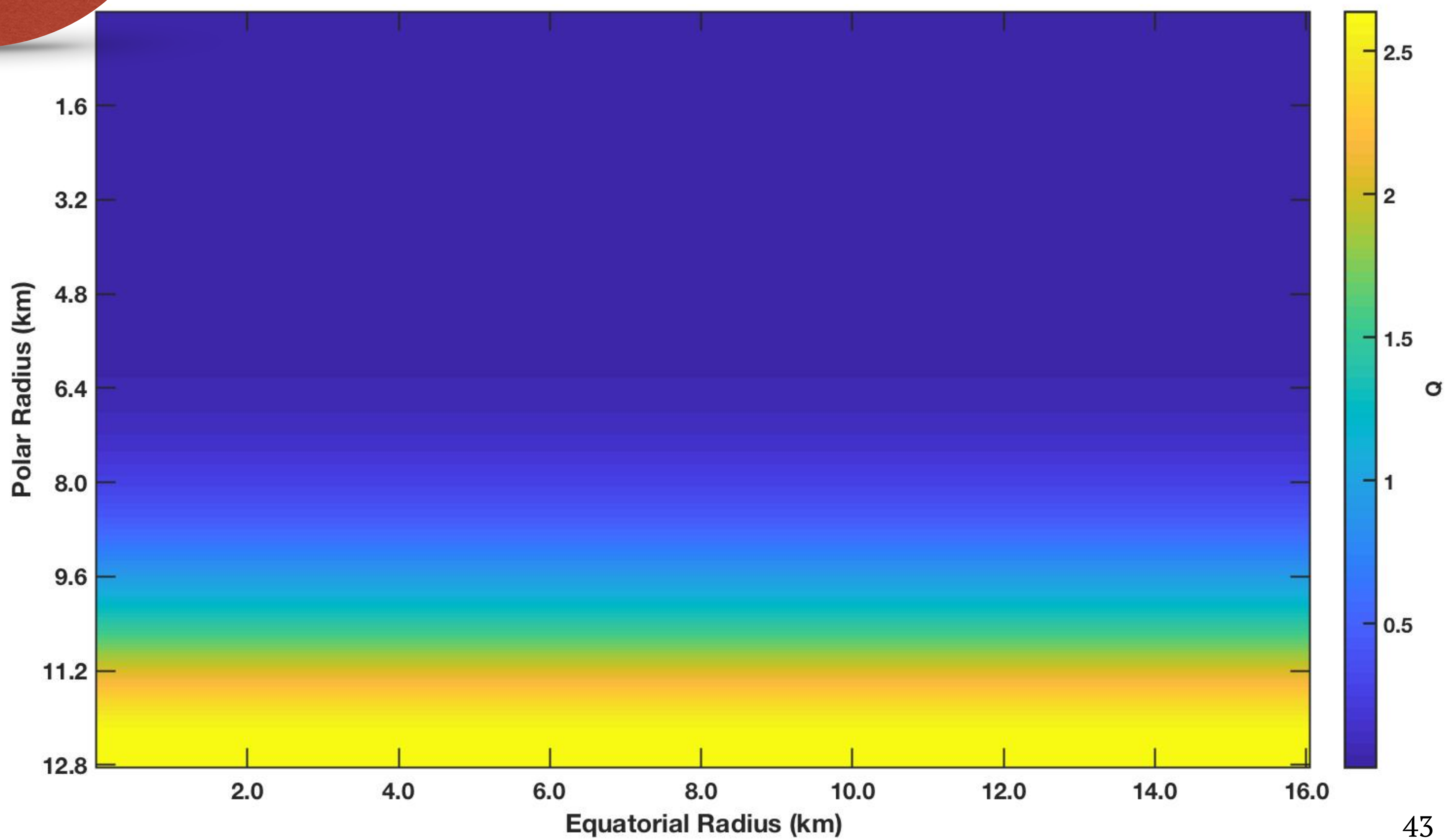


Model III

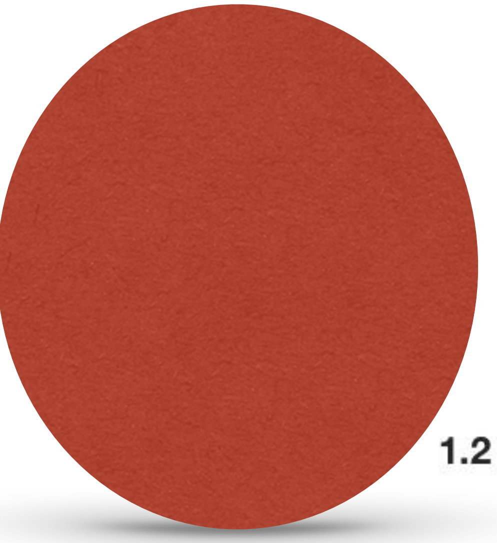
Mass Inhomogeneity — Oblate Stars (γ=0.8)



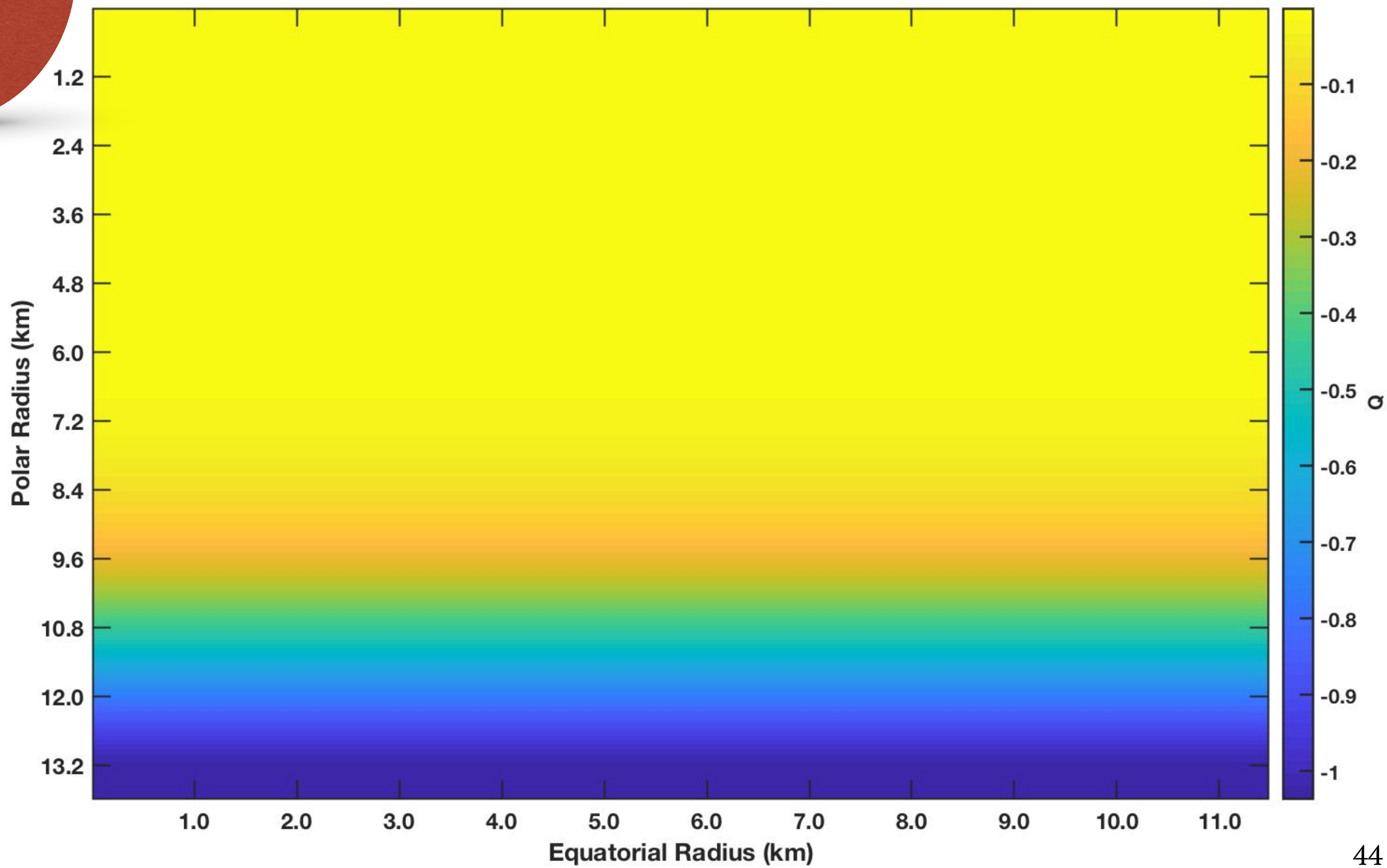
Model III



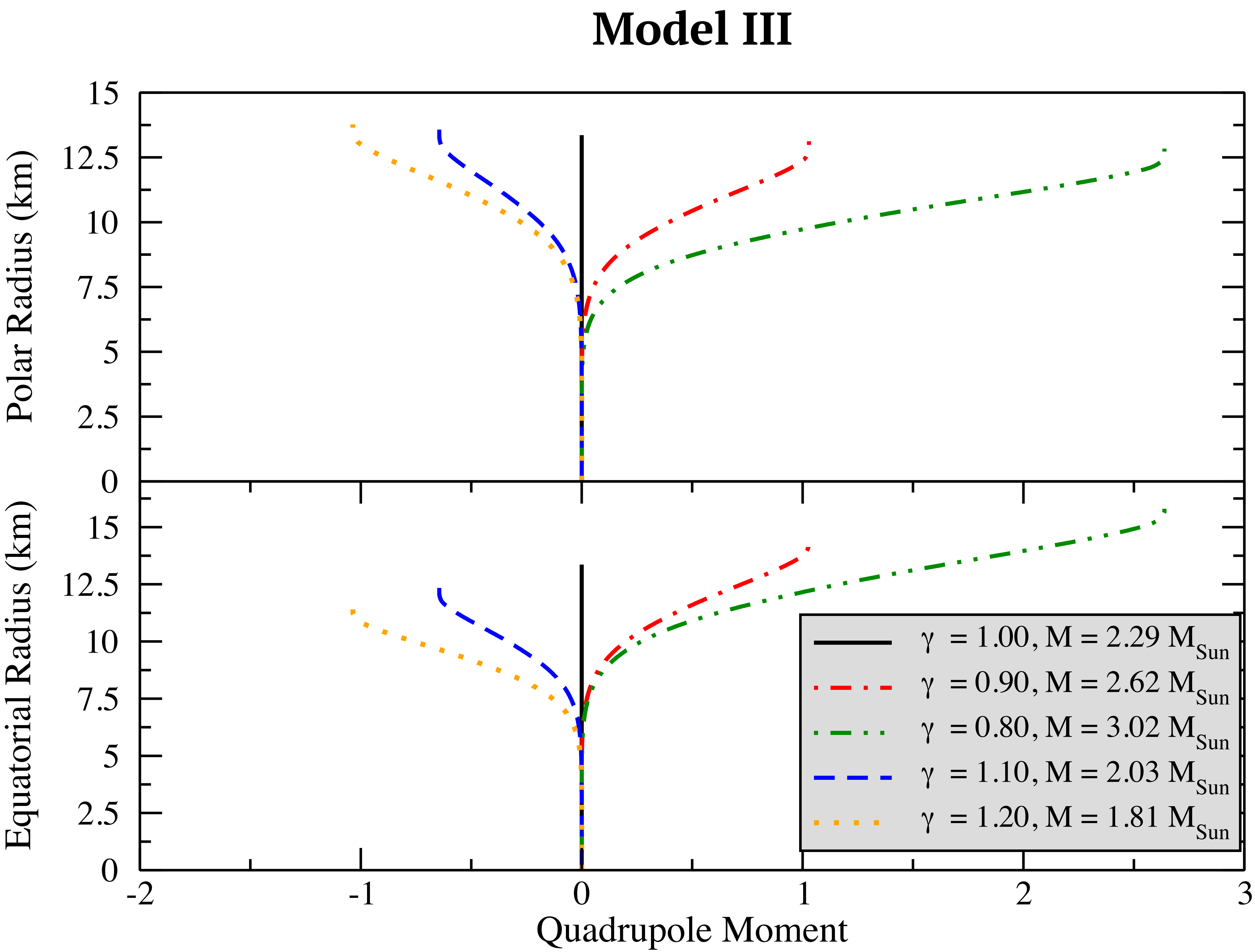
Mass Inhomogeneity   Prolate Stars (γ=1.20)



Model III

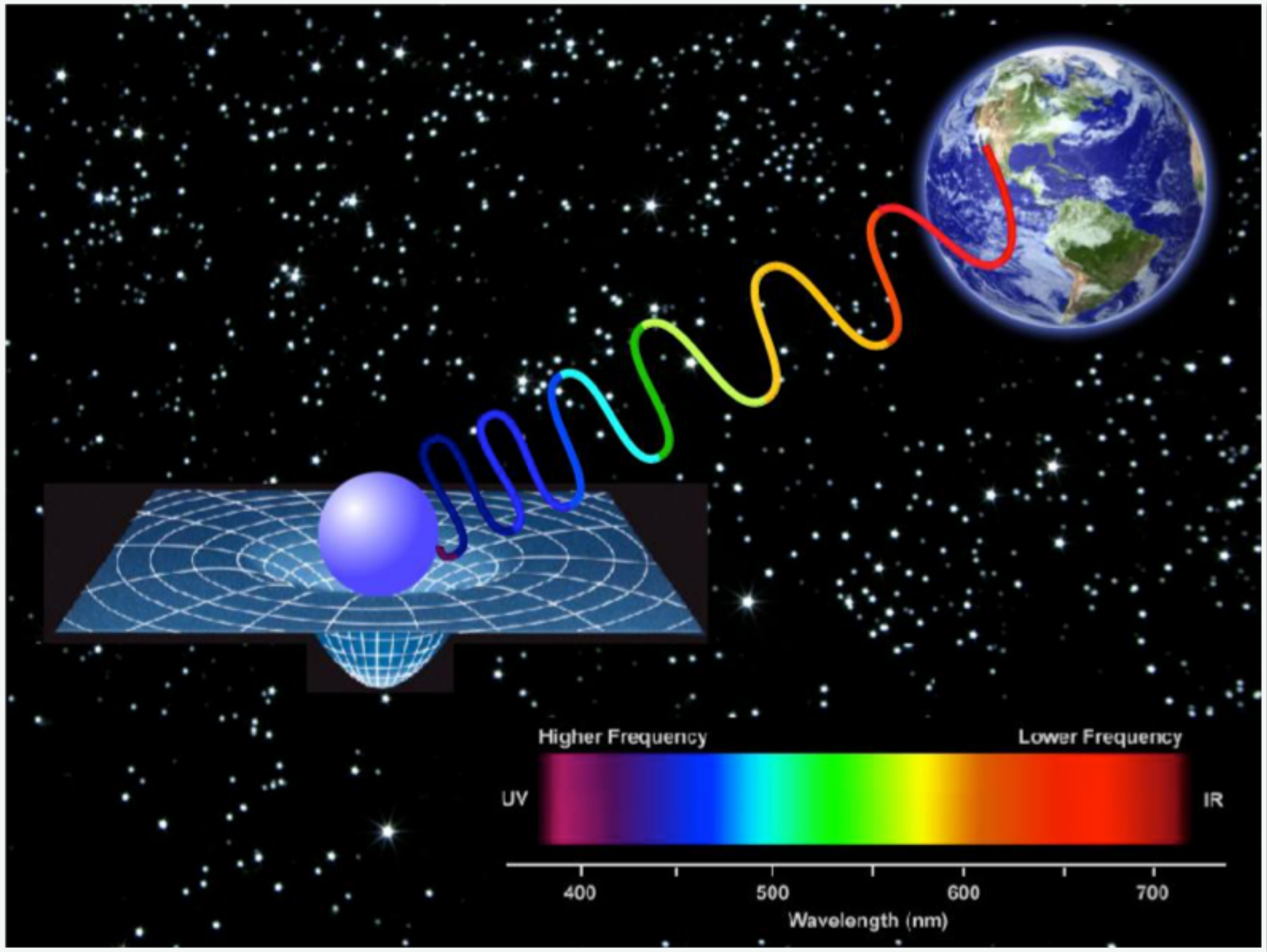


Mass Inhomogeneity



Gravitational Redshift

Gravitational Redshift



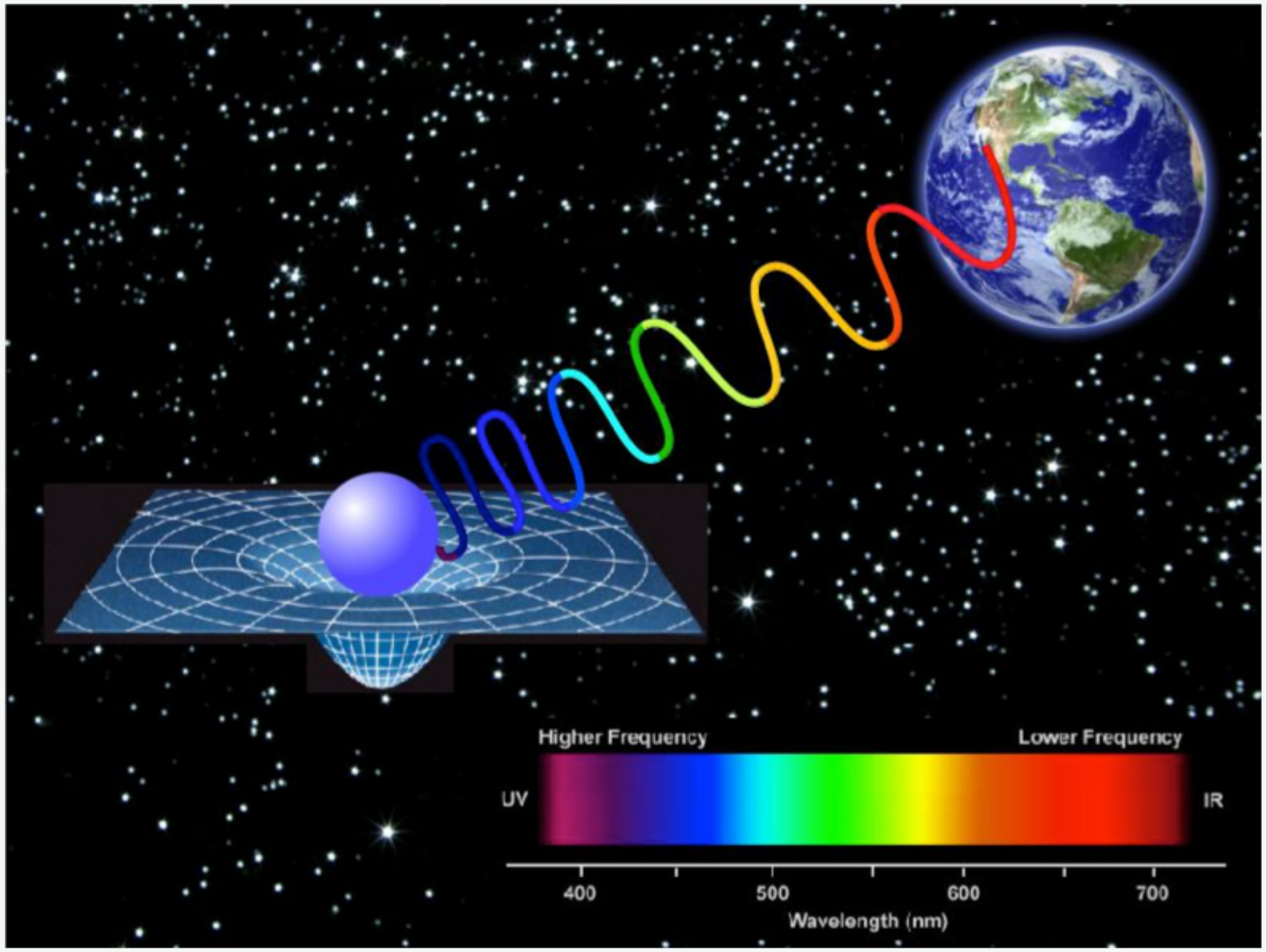
A. Romero (SDSU 2015)

$$z = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1$$

Gravitational Redshift

The gravitational redshift of deformed compact stars is governed by[1,2]:

$$z = \frac{1}{\left(1 - \frac{2M}{R}\right)^{\gamma/2}} - 1$$



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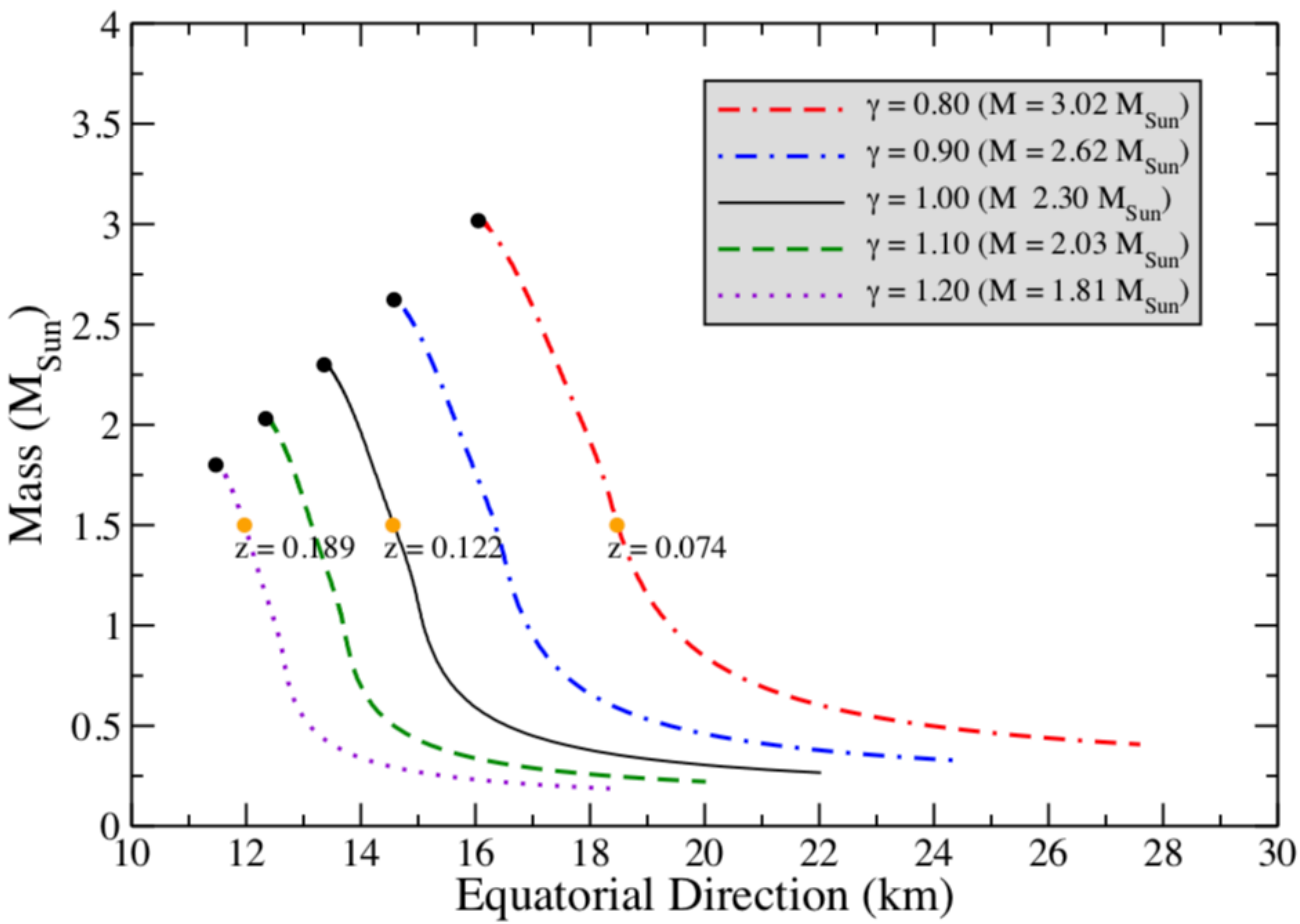
[1] O. Zubairi, A. Romero, and F. Weber, J. Phys. Conf.: Ser. **615**, 012003 (2015)
[2] O. Zubairi et al, Arxiv:1504-03006v1, (2015)

Gravitational Redshift

Model III



Zubairi et al, 2015

So Far...

- In this work:
  - Using our 1-D parameterized model, we were able to calculate the gravitational mass-quadrupole moment of non-rotating neutron stars.
  - Investigate the inhomogeneity of the mass distribution in oblate and prolate stars.
- From our results:
  - The mass distribution is ***not symmetric*** among oblate and prolate stars.
  - Hence, the deformation **does not** need to be high to see significant changes in stellar properties such as mass, radii, redshift, and quadrupole moment.
- Requires a more detailed description in 2-Dimensions…

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- Requires a more detailed description in 2-Dimensions...

2-D Stellar Structure

$$\frac{dP_{\parallel}}{dr} = \cdots \qquad \frac{dm_{\parallel}}{dr} = \cdots$$

$$\frac{dP_{\perp}}{dz} = \cdots \qquad \frac{dm_{\perp}}{dz} = \cdots$$

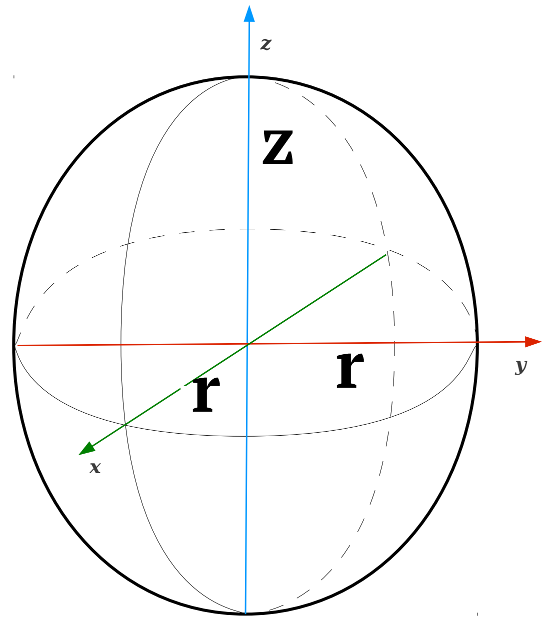
Use in conjunction with a *NON ISOTROPIC EoS*

Stellar Structure in 2D

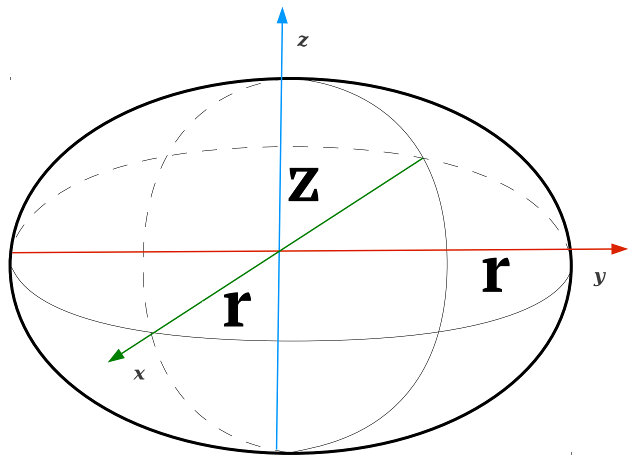
We need to look at the symmetry again…

Prolate spheroid with radii ***r*** & ***z***

Oblate spheroid with radii ***r*** & ***z***



Axial Symmetric



Weyl (*Weyl H. 1918*) Metric

$$ds^2 = \mathrm{e}^{2\Phi}\mathrm{dt}^2 - \mathrm{e}^{-2\Phi}\left[\mathrm{e}^{2\Lambda}\left(\mathrm{dr}^2 + \mathrm{dz}^2\right) + \mathrm{r}^2\mathrm{d}\phi^2\right]$$

Stellar Structure in 2D

$$ds^2 = \mathrm{e}^{2\lambda(r,z)}dt^2 - \mathrm{e}^{-2\lambda(r,z)}\left[\mathrm{e}^{2\nu(r,z)}\left(dr^2 + dz^2\right) + r^2 d\phi^2\right]$$

Need to calculate...

Stellar Structure in 2D

$$ds^2 = \mathrm{e}^{2\lambda(r,z)}dt^2 - \mathrm{e}^{-2\lambda(r,z)}\left[\mathrm{e}^{2\nu(r,z)}\left(dr^2 + dz^2\right) + r^2 d\phi^2\right]$$

Need to calculate...

Start with the Einstein Tensor

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$$

$$G^{t}{}_{t} \equiv R^{t}{}_{t} - \frac{1}{2}R$$

$$G^{r}{}_{r} \equiv R^{r}{}_{r} - \frac{1}{2}R$$

$$G^{r}{}_{z} \equiv R^{r}{}_{z}$$

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$$G^{z}{}_{z} \equiv R^{z}{}_{z} - \frac{1}{2}R$$

$$G^{\phi}{}_{\phi} \equiv R^{\phi}{}_{\phi} - \frac{1}{2}R$$

Stellar Structure in 2D

$$ds^2 = \mathrm{e}^{2\lambda(r,z)} dt^2 - \mathrm{e}^{-2\lambda(r,z)} \left[\mathrm{e}^{2\nu(r,z)} \left(dr^2 + dz^2 \right) + r^2 d\phi^2 \right]$$

Need to calculate...

Start with the Einstein Tensor

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$$

And equate to the Energy-Momentum Tensor

$$G^{\mu}{}_{\nu} = -8\pi T^{\mu}{}_{\nu}$$

$$T^{t}{}_{t} = \epsilon$$
$$T^{r}{}_{r} = P_{\parallel}$$
$$T^{z}{}_{r} = \tilde{P}$$
$$T^{r}{}_{z} = \tilde{P}$$
$$T^{z}{}_{z} = P_{\perp}$$
$$T^{\phi}{}_{\phi} = \tilde{P}$$

$$G^{t}{}_{t} \equiv R^{t}{}_{t} - \frac{1}{2} R$$
$$G^{r}{}_{r} \equiv R^{r}{}_{r} - \frac{1}{2} R$$
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$$G^{\phi}{}_{\phi} \equiv R^{\phi}{}_{\phi} - \frac{1}{2} R$$

Einstein's Field Equations

$$G^t{}_t = \frac{1}{r}\left[\mathrm{e}^{-2\nu+2\lambda}\left(2r\partial_r^2\lambda + 2r\partial_z^2\lambda + 2\partial_r\lambda - r\left(\partial_r\lambda\right)^2 - r\partial_z^2\nu \right.\right.$$
$$\left.\left. - r\partial_r^2\nu - r\left(\partial_z\lambda\right)^2\right)\right],$$

$$G^r{}_r = \frac{1}{r}\mathrm{e}^{-2\nu+2\lambda}\left(r\left(\partial_r\lambda\right)^2 - \partial_r\nu - r\left(\partial_z\lambda\right)^2\right),$$

$$G^z{}_r = G^r{}_z = \frac{1}{r}\mathrm{e}^{-2\nu+2\lambda}\left(\left(2r\partial_r\lambda\right)\left(\partial_z\lambda\right) - \partial_z\nu\right),$$

$$G^z{}_z = -\frac{1}{r}\mathrm{e}^{-2\nu+2\lambda}\left(r\left(\partial_r\lambda\right)^2 - \partial_r\nu - r\left(\partial_z\lambda\right)^2\right),$$

$$G^\phi{}_\phi = -\mathrm{e}^{-2\nu+2\lambda}\left(\left(\partial_r\lambda\right)^2 + \partial_z^2\nu + \partial_r^2\nu + \left(\partial_z\lambda\right)^2\right).$$

Zubairi et al, 2017

Einstein's Field Equations

Herrera (2013) calculated these components as well (&Negreiros 2018) and stated:

$$8\pi\mu = -\frac{1}{B^2}\left\{\frac{B''}{B} + \frac{D''}{D} + \frac{1}{r}(\frac{B'}{B} + \frac{D'}{D}) - (\frac{B'}{B})^2 + \frac{1}{r^2}\left[\frac{B_{\theta\theta}}{B} + \frac{D_{\theta\theta}}{D} - (\frac{B_\theta}{B})^2\right]\right\}, \quad (15)$$

$$8\pi P_{xx} = \frac{1}{B^2}\left[\frac{A'B'}{AB} + \frac{A'D'}{AD} + \frac{B'D'}{BD} + \frac{1}{r}(\frac{A'}{A} + \frac{D'}{D}) + \frac{1}{r^2}(\frac{A_{\theta\theta}}{A} + \frac{D_{\theta\theta}}{D} - \frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} - \frac{B_\theta D_\theta}{BD})\right], \quad (16)$$

$$8\pi P_{yy} = \frac{1}{B^2}\left[\frac{A''}{A} + \frac{D''}{D} - \frac{A'B'}{AB} + \frac{A'D'}{AD} - \frac{B'D'}{BD} + \frac{1}{r^2}(\frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} + \frac{B_\theta D_\theta}{BD})\right], \quad (17)$$

$$8\pi P_{zz} = \frac{1}{B^2}\left\{\frac{A''}{A} + \frac{B''}{B} - (\frac{B'}{B})^2 + \frac{1}{r}(\frac{A'}{A} + \frac{B'}{B}) + \frac{1}{r^2}\left[\frac{A_{\theta\theta}}{A} + \frac{B_{\theta\theta}}{B} - (\frac{B_\theta}{B})^2\right]\right\}, \quad (18)$$

$$8\pi P_{xy} = \frac{1}{B^2}\left\{\frac{1}{r}\left[-\frac{A'_\theta}{A} - \frac{D'_\theta}{D} + \frac{B_\theta}{B}\left(\frac{A'}{A} + \frac{D'}{D}\right) + \frac{B'}{B}\frac{A_\theta}{A} + \frac{B'}{B}\frac{D_\theta}{D}\right] + \frac{1}{r^2}(\frac{A_\theta}{A} + \frac{D_\theta}{D})\right\}, \quad (19)$$

$$P'_{xx} + \frac{A'}{A}(\mu + P_{xx}) + \frac{B'}{B}(P_{xx} - P_{yy}) + \frac{D'}{D}(P_{xx} - P_{zz})$$
$$+\frac{1}{r}\left[\left(\frac{A_\theta}{A} + 2\frac{B_\theta}{B} + \frac{D_\theta}{D}\right)P_{xy} + P_{xy,\theta} + P_{xx} - P_{yy}\right] = 0,$$

$$P_{yy,\theta} + \frac{A_\theta}{A}(\mu + P_{yy}) + \frac{B_\theta}{B}(P_{yy} - P_{xx}) + \frac{D_\theta}{D}(P_{yy} - P_{zz}) + r\left[\left(\frac{A'}{A} + 2\frac{B'}{B} + \frac{D'}{D}\right)P_{xy} + P'_{xy}\right] + 2P_{xy} = 0.$$

Herrera et al, arXiv:1301.2424 [gr-qc], 2013 & Negreiros et al, Universe, 2018

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Einstein's Field Equations

Since the mathematical form of the Weyl metric, is axial symmetric…

$$ds^2 = \mathrm{e}^{2\lambda(r,z)} dt^2 - \mathrm{e}^{-2\lambda(r,z)} \left[\mathrm{e}^{2\nu(r,z)} \left(dr^2 + dz^2 \right) + r^2 d\phi^2 \right]$$

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in the energy-momentum tensor as described by

$$T^\mu{}_\lambda = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_{||} & \tilde{P} & 0 \\ 0 & \tilde{P} & P_\perp & 0 \\ 0 & 0 & 0 & \tilde{P} \end{pmatrix},$$

Stellar Structure in 2D

$$ds^2 = \mathrm{e}^{2\lambda(r,z)}dt^2 - \mathrm{e}^{-2\lambda(r,z)}\left[\mathrm{e}^{2\nu(r,z)}\left(dr^2 + dz^2\right) + r^2 d\phi^2\right]$$

Due to cylindrical symmetry, we need to consider the mass in both the **radial** and **polar** directions.

Stellar Structure in 2D

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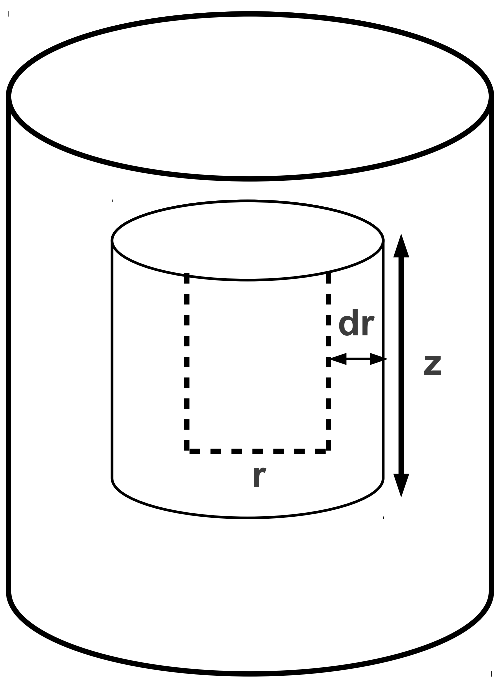
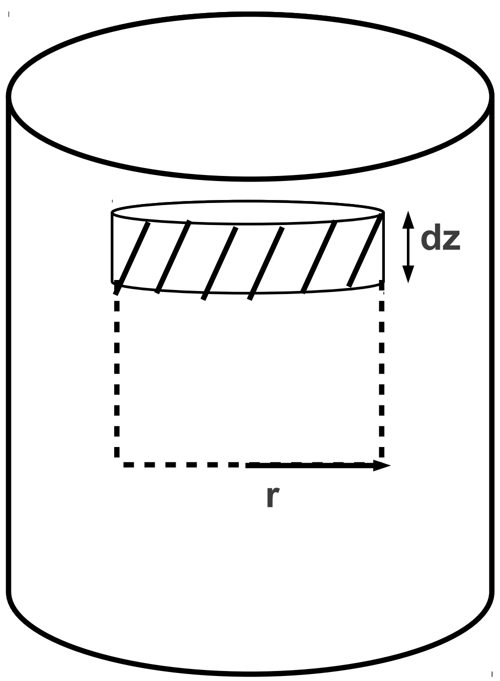
We define *two* differential masses…





radius ***r***, height ***z***, thickness ***dr***

$$\frac{dm}{dr} = \epsilon 2\pi r z$$

slab of radius ***r*** and thickness ***dz***

$$\frac{dm}{dz} = \epsilon \pi r^2$$

Stellar Structure in 2D

Thus, with our differential masses:

$$\partial_r m(r,z) = 2\pi r z \epsilon(r,z)\ ,$$

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Stellar Structure in 2D

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$$\partial_r m(r,z) = \frac{rz}{4} G^t{}_t\,.$$

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The ***r*** term vanishes!!!

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Stellar Structure in 2D

Starting with our γ-TOV...

$$\frac{dP}{dr} = -(\epsilon + P)\frac{\left[\frac{1}{2}r + 4\pi r^3 P - \frac{1}{2}r\left(1 - \frac{2m}{r}\right)^{\gamma}\right]}{r^2\left(1 - \frac{2m}{r}\right)^{\gamma}}$$

along with our parameterization: **$z=\gamma r$**, we can apply a transformation...

Stellar Structure in 2D

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Our step ***dr*** is small (on the order of meters) ⟶ $\gamma = \frac{dz}{dr}$

Stellar Structure in 2D

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Stellar Structure in 2D

Starting with our γ-TOV…

$$\frac{dP}{dr} = -(\epsilon + P)\frac{\left[\frac{1}{2}r + 4\pi r^3 P - \frac{1}{2}r\left(1 - \frac{2m}{r}\right)^\gamma\right]}{r^2\left(1 - \frac{2m}{r}\right)^\gamma}$$

along with our parameterization: **z=γr**, we can apply a transformation…

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$$\frac{dP_{\parallel}}{dr} = -(\epsilon + P_{\parallel})\frac{\left[\frac{1}{2}r + 4\pi r^3 P_{\parallel} - \frac{1}{2}r\left(1 - \frac{2m(r,z)}{r}\right)^\gamma\right]}{r^2\left(1 - \frac{2m(r,z)}{r}\right)^\gamma}$$

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2-D Stellar Structure[3,4]

We need to break up the mass equally to take the mass in the equatorial and polar directions separately but that are confined in a total gravitational mass given by:

$$\mathcal{M}(r,z) = 2\pi r z\epsilon + \pi r^2\epsilon - \frac{1}{3}\pi r^2 z\epsilon$$

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[3] O. Zubairi, and F. Weber, J. Phys. Conf.: Ser. **845**, 012005 (2017)
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$P_{\parallel}(r=R)=0$

$P_{\perp}(z=Z)=0$

$\gamma=1$ → The deformation is completely ***dictated*** by the anisotropies in the EoS…

$$\frac{dP_{\parallel}}{dr} = -\frac{(\epsilon+P_{\parallel})\left[\frac{1}{2}r + 4\pi r^3 P_{\parallel} - \frac{1}{2}r\left(1-\frac{2\mathcal{M}(r,z)}{r}\right)^{\gamma}\right]}{r^2\left(1-\frac{2\mathcal{M}(r,z)}{r}\right)^{\gamma}}$$

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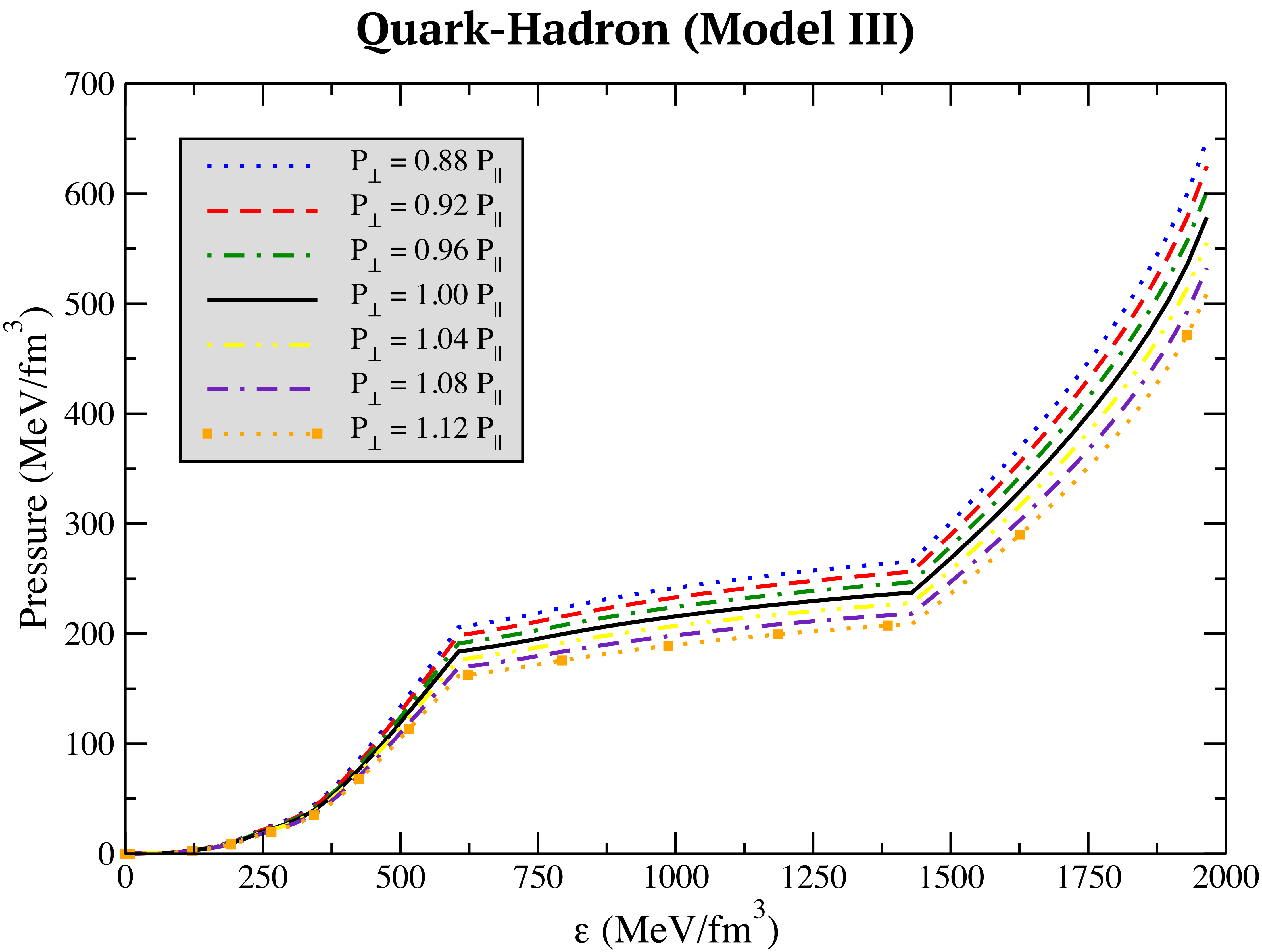
Anisotropic Eos Model[3]



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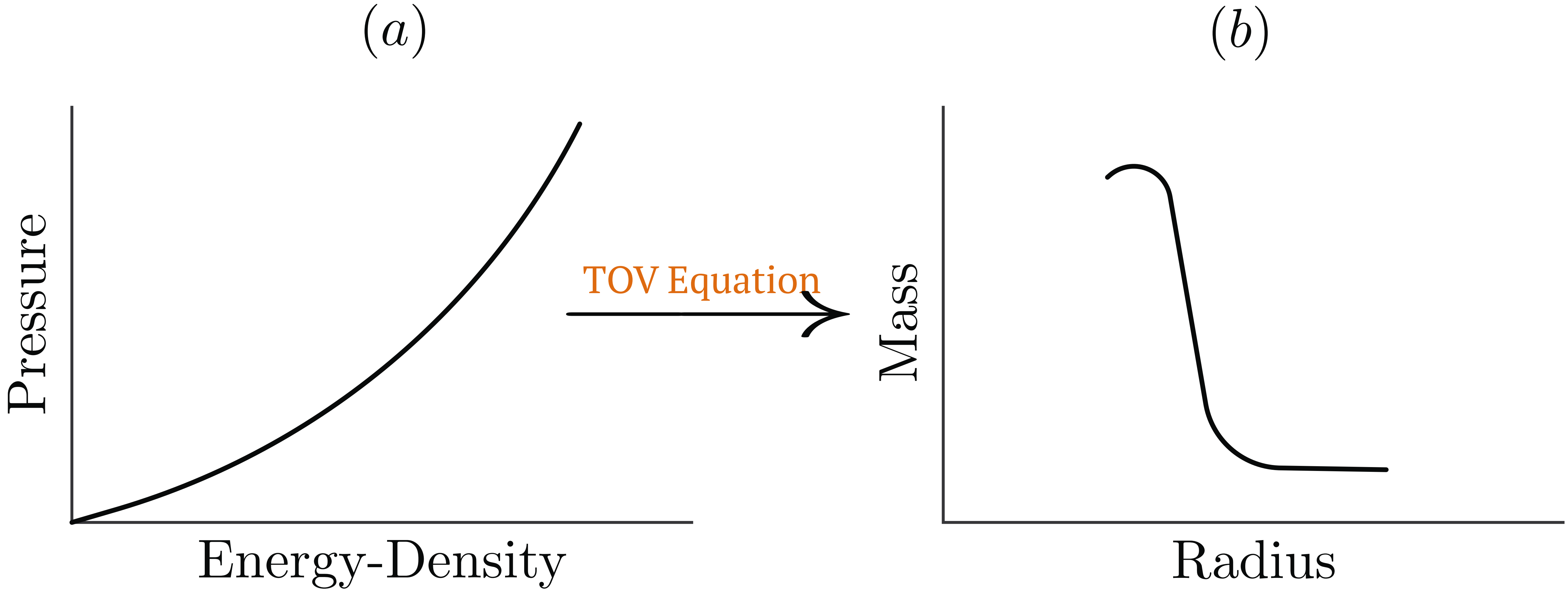

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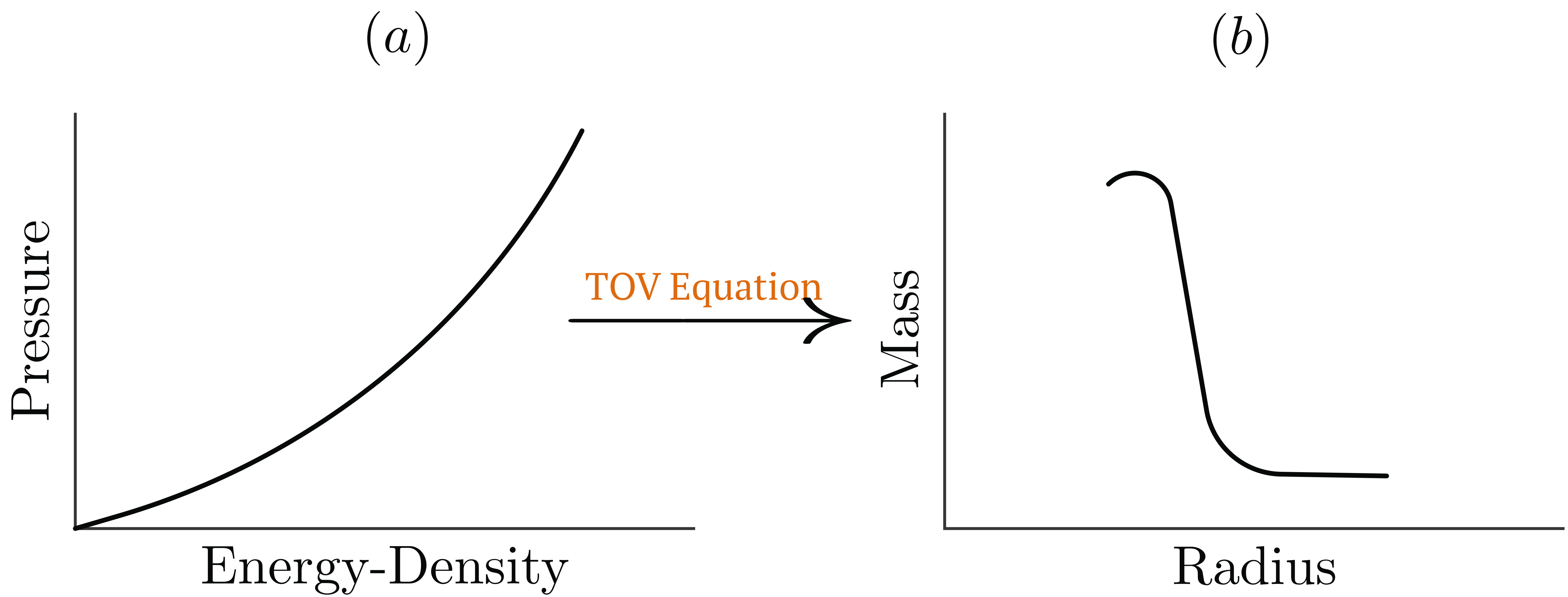

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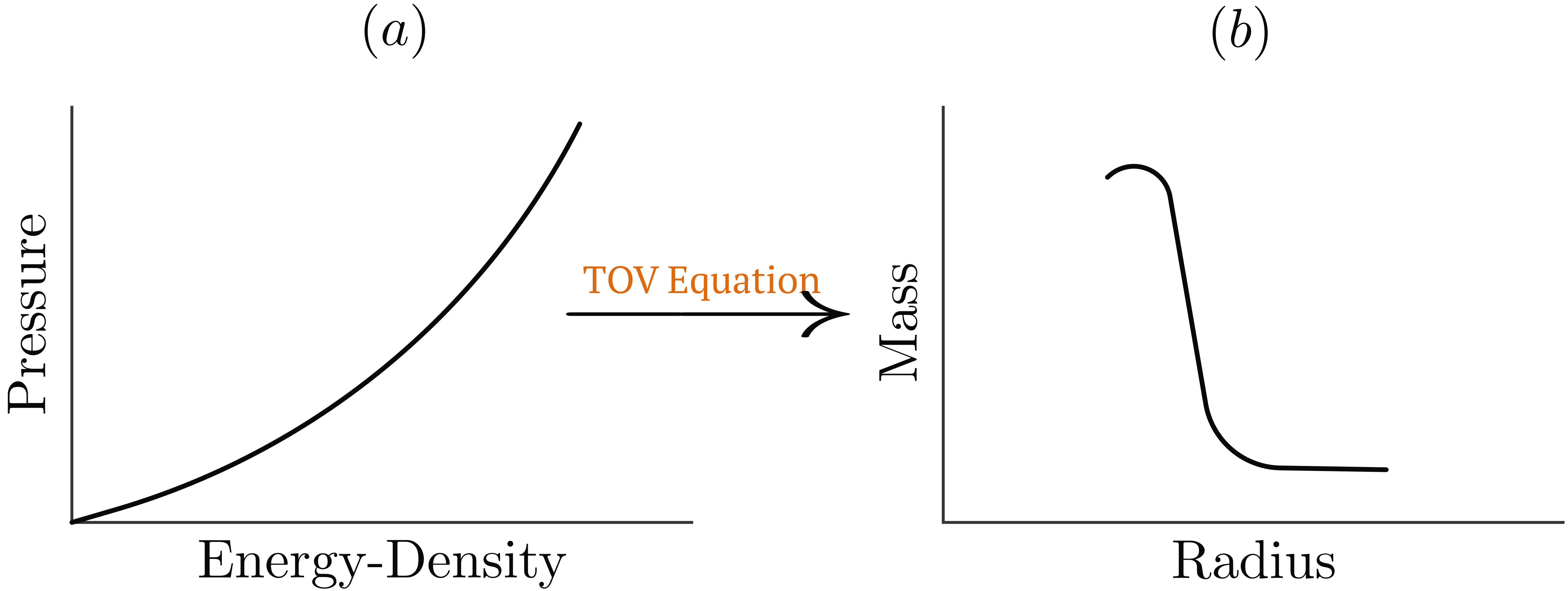

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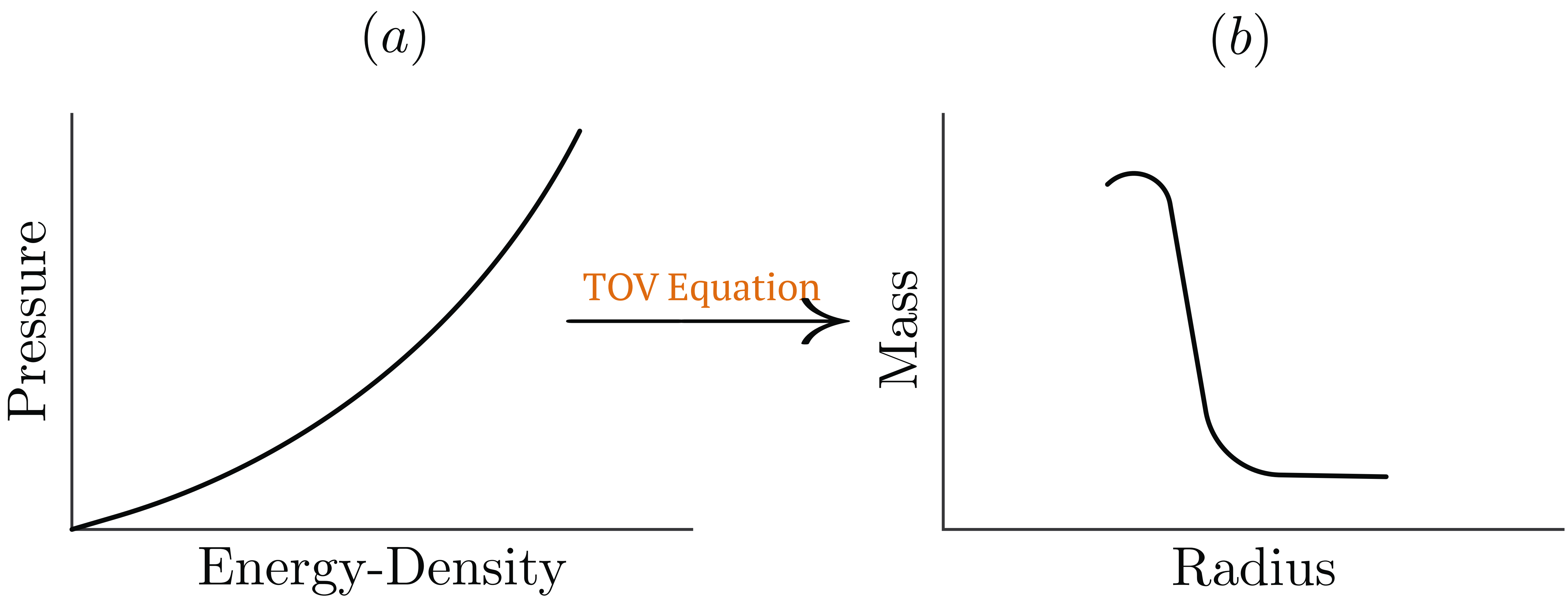

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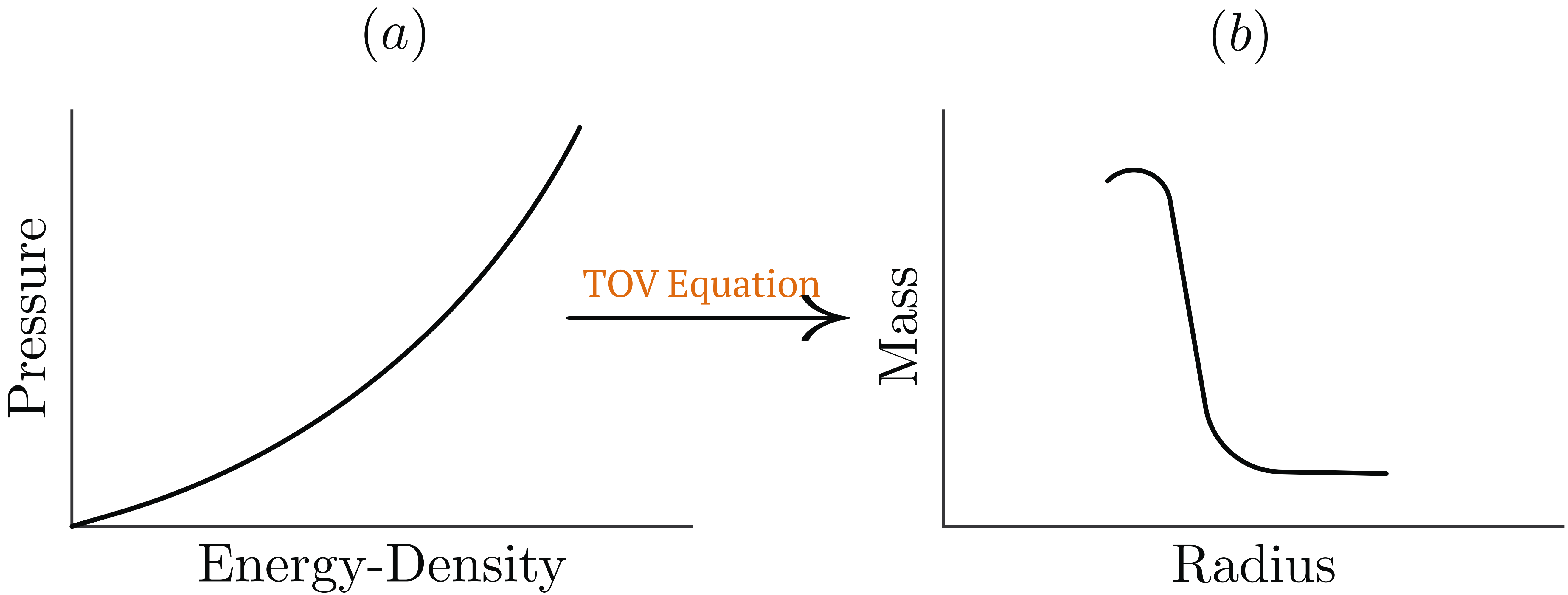
- We can not simply have mass-radius relations—as they will not give us any information on the deformation.
- We will have to look at the internal properties—such as pressure profiles ***both*** in the equatorial and polar directions.
- From the non-isotropic EoS, we can then investigate the change in mass.



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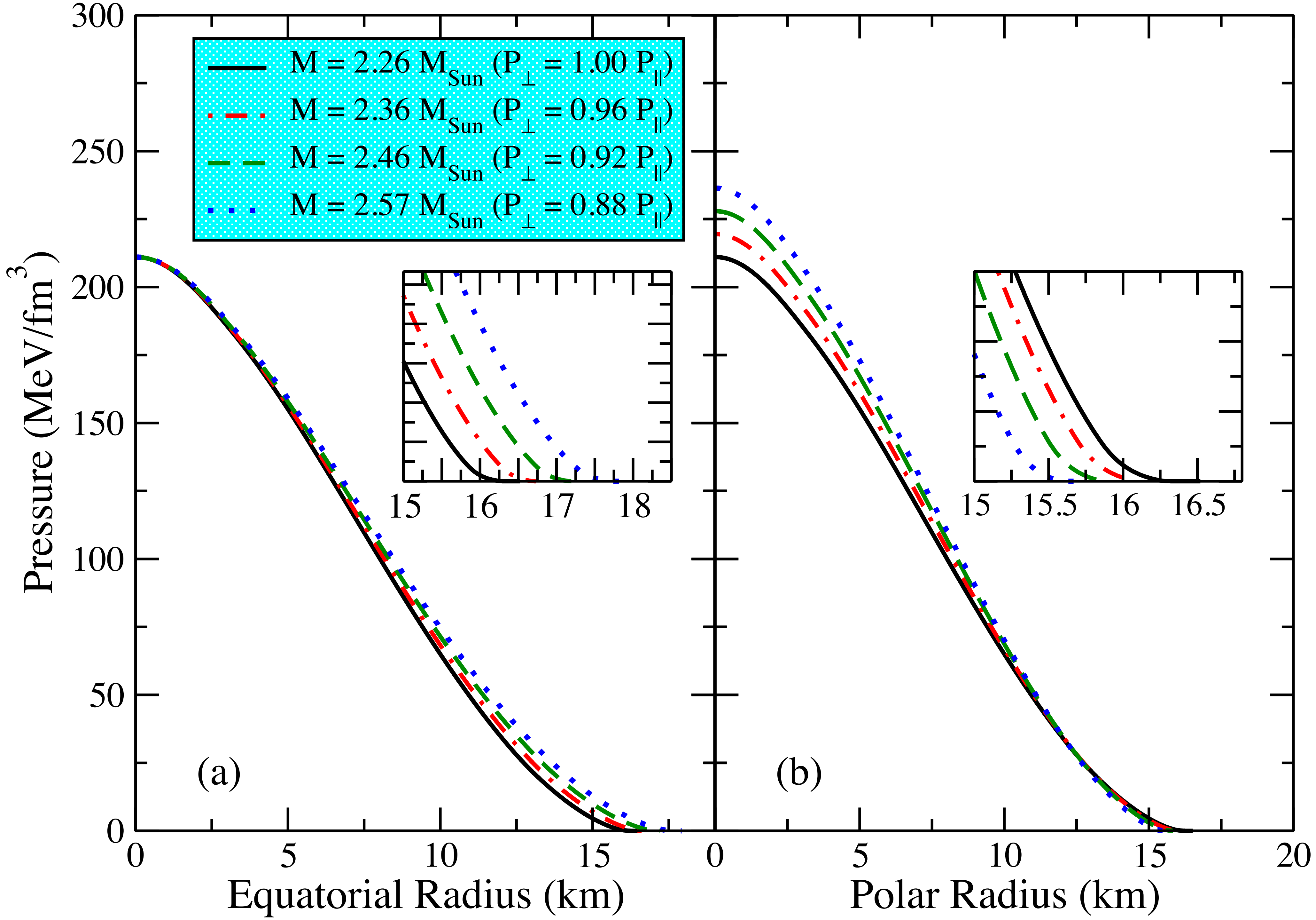
Pressure Profiles—2D Model[3,4]

Oblate Stars



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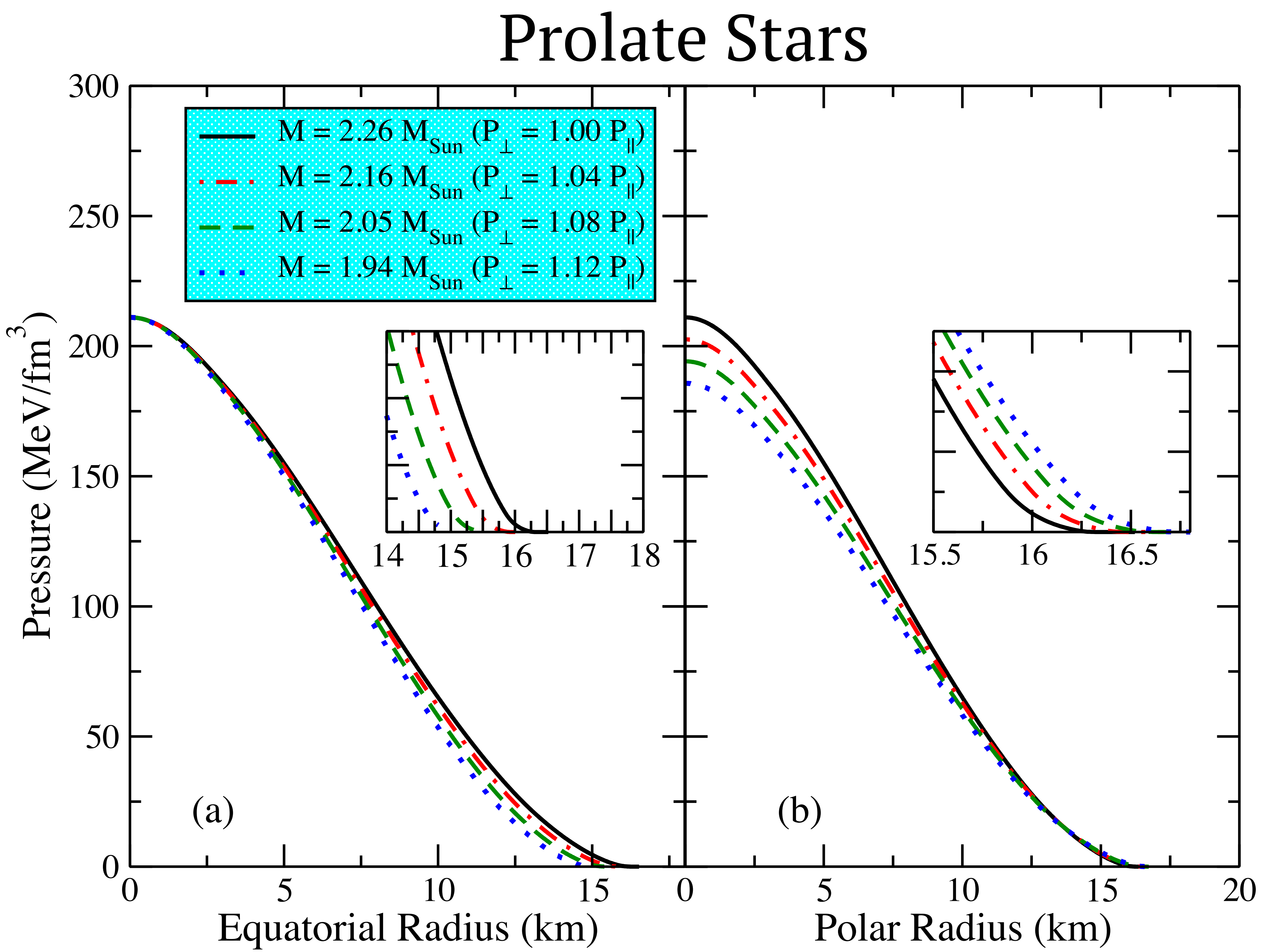

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Results—2D Model

Mass (M⊙)	R (km)	Z (km)	Quadrupole Moment	Shape
2.57	17.92	15.77	1.1231	Oblate
2.46	17.21	15.83	0.7026	Oblate
2.36	16.73	16.06	0.3301	Oblate
2.26	16.51	16.51	0	Sphere
2.16	16.00	16.67	-0.2993	Prolate
2.05	15.39	16.73	-0.5673	Prolate
1.94	14.80	16.82	-0.8065	Prolate

Conclusions

- In this work:
  - Using our 1-D parameterized model and our 2-D model, we were able to calculate the gravitational mass-quadrupole moment of non-rotating neutron stars.
  - Investigate the inhomogeneity of the mass distribution in oblate and prolate stars.
- From our results:
  - The mass distribution is ***not symmetric*** among oblate and prolate stars.
  - Thus, these deformed objects are distinct and are **NOT** the same.
  - From the stellar properties such as **masses**, **radii**, **pressure and energy-density profiles**, **gravitational redshift**, and **quadrupole moment**, we see that deformation plays a pivotal role in the stellar structure in of these compact objects…
  - Hence, the deformation **does not** need to be high to see significant changes in said stellar properties.
- Continue this work by analyzing the Weyl metric in greater detail (i.e. perturbations)
- Which will (hopefully) allow unique solutions to axially symmetric geometries for stellar configurations.

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Thank You!

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Student Involvement:

- Deanna Kondek (2017-Present—WIT)
- Megi Baliko (2017-Present—WIT)
- *Wei Liang (2017-Present—WIT)*
- Ryan Maresca (2017-Present—WIT)
- *Greg Shao (Summer 2017–WIT)*
- *David Wigley (2016-Present—WIT)*
- Temour Raza (2016-Present—WIT)
- Scott Moir (Spring 2017—WIT)
- *Alexis Romero (2014-2015–SDSU)*

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